Six Ideas That Shaped Physics

Chapters Q9a and Q9b

(Replaces Q6–Q9)

Q9a The Wavefunction

Chapter Overview

Introduction

In chapter Q5, we saw that experiments involving quantons of all types showed that probability patterns evolve in a wavelike manner, but when we seek to locate quantons, they seem like particles. In this chapter, we will explore how we might describe this behavior mathematically.

Section Q9a.1: The Wavefunction Concept

In order to explain what we observe in a quanton-at-a-time two slit interference experiment, we must assume that each individual quanton is fundamentally described by a wave, which we call its **quantum wavefunction** $\psi(t, x, y, z)$. The **time-dependent Schrödinger equation** describes how this wavefunction evolves with time, but for our purposes, we need only know that it behaves qualitatively like the mechanical waves with which we are familiar. Because the quantum wavefunction also obeys the **superposition principle**, quantum wavefunctions moving through two slits create the same kind of interference patterns we see with other waves. The probability of locating a quanton at a given location and at a given time is proportional to the square of its wavefunction's value at that location and time.

But unless we actively determine a quanton's location, it does not have a location. A quanton is not a classical particle that follows a trajectory. It is more like the twist in a Möbius strip that has no well-defined "location" along the strip until we constrain it to a given location by making the rest of the strip flat.

Because the quantum wavefunction at a given time determines the results of future experiments (at least probabilistically) in a given environment, we say that the wavefunction describes the quanton's **quantum state** at that time. (We would describe corresponding state of a *classical* particle in a given environment by specifying its position $\vec{r}$ and momentum $\vec{p}$ at that time.)

At an instant of time, quantum wavefunctions are generally functions $\psi(x, y, z)$ of the three spatial coordinates and have values that are complex numbers. In this text, we will (unless otherwise specified) consider quantons confined to move in one dimension (taken to be the x axis). Also, we will (outside of appendix QA) consider only situations where we can consider $\psi(x)$ to be real.

Section Q9a.2: Wavefunctions and Position Probability

If a quanton's wavefunction is $\psi(x)$ and we do an experiment to determine the quanton's position, the probability of any specific outcome x_i is proportional to $|\psi(x_i)|^2$. (Using the absolute value bars is notation that is conventional and necessary when the wavefunction's value is a complex number, and so worth getting used to.) But even in one dimension, we have an infinite number of possible positions, so the probability of being at any one mathematical position would have to be zero. We get around this problem by considering a tiny range of positions from x_i to $x_i + dx$ and defining the probability for this *range* to be proportional to $|\psi(x_i)|^2 dx$, where dx is small enough that $\psi(x)$ is approximately constant over the range. We can get the probability for a larger range by dividing the larger range up into tiny steps and adding up the probabilities for each tiny range. We can calculate this using an integral:

$$\Pr(x_1 \ to \ x_2) = \frac{\int_{x_1}^{x_2} |\psi(x)|^2 dx}{\int_{-\infty}^{+\infty} |\psi(x)|^2 dx} \qquad \text{(Q9a.4)}$$

$$= \int_{x_1}^{x_2} |\psi(x)|^2 dx \ \text{ when } \ 1 = \int_{-\infty}^{\infty} |\psi(x)|^2 dx \qquad \text{(Q9a.6)}$$

- **Purpose:** This equation links a quanton's wavefunction $\psi(x)$ and the probability that quanton's position (when measured) is found to be in the range x_1 to x_2.
- **Limitations:** This equation applies only to quantons confined to move in only one dimension (which we take to be the x axis).

Dividing by $\int_{-\infty}^{+\infty} |\psi(x)|^2 dx$ ensures that the total probability of finding the quanton to be *somewhere* is 1, no matter how we actually scale the wavefunction. However, we can conveniently choose to **normalize** the wavefunction by scaling it (multiplying it by an overall constant) so that $1 = \int_{-\infty}^{+\infty} |\psi(x)|^2 dx$, allowing us to use the simpler equation Q9a.6. However, note that only wavefunctions where $\int_{-\infty}^{+\infty} |\psi(x)|^2 dx$ is a finite number can possibly be physically valid quantum wavefunctions.

Several examples in this section illustrate applications of these equations.

Section Q9a.3: Eigenfunctions and Eigenvalues

A quanton whose state is described by a general wavefunction does not have a well-defined position, but if the quanton's wavefunction is a very thin spike at a certain location x_0, then the quanton's position *is* well-defined, because if we measure the positions of quantons in that state, we will get x_0 with certainty. We say that a spike at x_0 is a **eigenfunction** of position and that x_0 is the function's corresponding **eigenvalue**. Spikes at different locations correspond to different position eigenfunctions.

The de Broglie relation implies that a wavefunction having a definite wavelength λ_0 (such as a continuous one-dimensional sine wave) will be an eigenfunction of x-momentum with the corresponding eigenvalue $p_{x0} = h/\lambda_0$, because measuring the x-momentum of a quanton having that wavefunction will yield p_{x0} with certainty. We can also find special wavefunctions that certainly yield specific values of quantities like energy and orbital angular momentum. We define an **observable** to be a quantity (such as position, momentum, or energy) whose measurement results might vary for a given type of quanton. Any wavefunction that *certainly* yields a particular value a_0 when we measure an observable A is an eigenfunction of A with eigenvalue a_0.

Section Q9a.4: The Heisenberg Uncertainty Principle

Position and momentum eigenfunctions are incompatible: a wavefunction corresponding to a well-defined x-momentum (an infinitely long sine wave) looks nothing like a wavefunction with a well-defined x-position (a spike) and vice versa. This means that a quanton's wavefunction can never be such that its position and momentum are simultaneously well-defined.

Indeed, we can analyze a sine wave confined to a certain length to show that its implied x-position and x-momentum uncertainties Δx and Δp_x are inversely proportional, with the constant of proportionality being some fraction of Planck's constant h. A more general mathematical proof shows that for all possible wavefunctions, $\Delta p_x \Delta x \geq \frac{1}{2}\hbar$, where $\hbar \equiv h/2\pi$. This is the famous **Heisenberg Uncertainty Principle**.

In the latter statement of the principle, Δp_x is the standard deviation of the results of measurements of x-momentum performed on many quantons described by a given wavefunction, and Δx is the standard deviation of results from experiments to determine x-position performed on many (other) quantons described by that same wavefunction. The principle is therefore a *statistical* statement about possible experimental outcomes. (Some popular treatments of the principle blur this important aspect of its meaning, attempting to apply it to a single quanton.) Note also that the principle is *not* a fundamental assumption about quantum mechanics, but rather a *consequence* of the de Broglie relation and the way we have defined the wavefunction.

Q9a.1 The Wavefunction Concept

Our goal in this chapter is to develop a model for quantitatively modeling the strange behavior of quantons described in chapter Q5. This model will provide the foundation for a new theory of how quantons move and interact that we call **quantum mechanics**.

A review of what we need to explain

Let's review what this model must explain. In section Q5.6, we saw that when we send quantons one at a time through a two-slit interference experiment, we find that when we seek to *locate* quantons, we find that they have precise and specific locations, as if they were particles. But we also saw that the *probability patterns* that describe where quantons are more or less likely to be found are described well by a wave model. Since the interference pattern emerges even when we send the quantons through the experiment one at a time, the wave that predicts the probability pattern must be associated with *each individual quanton* rather than the collective set of quantons involved in the experiment. Finally, we also saw that using proximity detectors to show which slit a given quanton goes through does show them going through one slit or the other (not both at the same time), but doing this *ruins* the interference pattern, yielding a probability pattern that is the sum of two single-slit probability patterns, not a double-slit interference pattern. These are the experimental facts that our model of quantum mechanics must explain.

Quantum wavefunctions are the foundation

These experimental facts strongly encourage us to suppose that a quanton is fundamentally described by some kind of wave, which we will call its **quantum wavefunction** $\psi(t,x,y,z)$, a function of both position and time. This wavefunction must evolve in time according to an equation similar to those describing mechanical waves such as sound waves or water waves. In this text, we will not explore the detailed character of this equation (which physicists call the **time-dependent Schrödinger equation**). For our purposes, we need only know that quantum wavefunctions satisfying this equation behave qualitatively like the mechanical waves we have already studied. We also need to suppose that quantum wavefunctions obey the **superposition principle**, so that quantum waves emerging from two slits add constructively or destructively like mechanical waves do. If quantum wavefunctions satisfy these requirements, then sending quantons through a two-slit interference apparatus will generate the same kind of interference pattern in each quanton's wavefunction that we observe with mechanical waves.

Position probability is proportional to the *square* of the quantum wavefunction

The experimental facts also constrain us to suppose that the quantum wavefunction at a given point must be linked the *probability* of finding the quanton at that point. But the probability cannot be *directly* proportional to the wavefunction's value at that point, because waves typically oscillate between positive and negative values, and negative probabilities are absurd. The behavior of classical waves gives us a clue for how to proceed. In an interference pattern generated by sound waves, the sound's loudness (or more precisely, the sound's average **intensity** $\equiv$ power delivered per unit area) at a point turns out to be proportional to the *square* of the sound wave's value at that point. This is also true for light considered as a classical wave. But from a quantum viewpoint, light's intensity at a photodetector is proportional to the number of photons per second that detector receives, which in turn is proportional to the probability of photons landing on it. So everything will work for light if we assume that the probability of finding a photon at a given position is proportional to the *square* of its wavefunction at that position (something that will automatically be a positive number). We will assume this statement applies to all other kinds of quantons as well.

Quantons generally have no well-defined location

In this model, we do assume anything about "where the quanton is" inside this wavefunction. Such a question is at best meaningless, and can be counterproductive (for example, as one struggles to visualize the quanton as a single particle going through both slits at once). It is better to say that the quanton *has no meaningful position* until a detector registers it.

An analogy

Here is an analogy. Imagine twisting a strip of paper by one half-turn and gluing its ends together to create a Möbius strip. Though this strip unambiguously has a twist of exactly one half-turn *somewhere* along it, where exactly is that twist located? This is *generally* a meaningless question: the twist is a feature of the strip as a whole and cannot be assigned a specific position (see figure Q9a.1a). But we can forcibly *confine* the twist by laying the strip on a table (with its bottom side lying flat on the table) and then pressing the strip's top side so that it lies flat on top of the bottom side. As we constrain more and more of the strip's top side to become flat, the twist becomes more and more confined to a limited range (the part of the strip that we can't make flat: see figure Q9a.1b). If the strip were infinitely stretchable, we could confine its twist to an infinitesimal range, defining its position precisely.

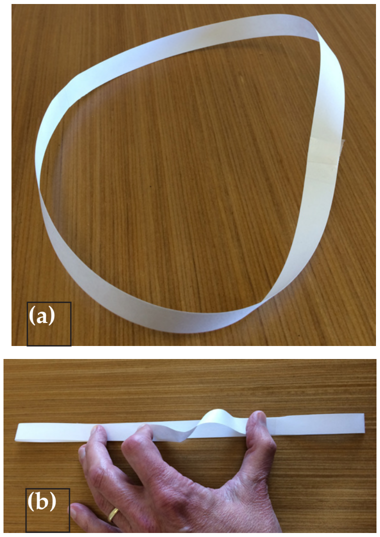


Figure Q9a.1
(a) Where exactly is the half-twist in this Möbius strip? (b) We can force the twist's location to be more precisely defined.

A quanton described by a quantum wavefunction is analogous. Generally, we can say that a wavefunction describes exactly one quanton, but that quanton has no meaningful location. Detecting a quanton at a specific location is analogous to forcibly locating the strip's twist by flattening other parts of the strip until the twist has a well-defined location. The process of detection essentially *defines* the quanton's previously undefined location this way.

This model is very different from the classical picture of how the universe operates. In Newtonian mechanics, the universe consists of *particles* that follow well-defined trajectories through space and time. *In principle,* if we know all the forces that act on a particle and the particle's initial position $\vec{r}$ and velocity $\vec{v}$ (or equivalently, its initial position $\vec{r}$ and momentum $\vec{p}$), we can predict its future trajectory. Knowing a particle's $\vec{r}$ and $\vec{p}$ at an instant therefore defines its physical **state** at that instant: in a given environment, that is all you need to know to predict the particle's entire future.

A quanton's wavefunction describes its "state"

In the quantum model, the universe consists of quantons described by wavefunctions. While quantons are countable (like the half-twists on a strip are countable) they generally have no meaningful locations, and therefore no meaningful trajectories. We can predict a quanton's future location in a given environment only probabilistically. But just as knowing a classical particle's position and momentum at a given time in a given environment (in principle) determines its position and momentum at all future times, knowing a quanton's wavefunction at a given time in a given environment (in principle) determines its wavefunction (and thus position probabilities) at future times, which is the most that we can know about a quanton's position. A quanton's wavefunction turns out to predict (probabilistic) results for future experiments that measure other quantities as well. We therefore call a quanton's wavefunction $\psi(x,y,z)$ at a given instant its **quantum state** at that instant, analogous to the state $\vec{r}$ and $\vec{p}$ of a classical particle.

We will generally consider quantons moving in only one dimension

Quantum wavefunctions (at a given instant of time) are generally functions of position in three dimensional space. However, functions of three variables are hard to visualize and mathematically difficult. We can learn much of what we need to know about wavefunctions by considering quantons confined to move in along a single axis, which we conventionally take to be the x axis. We can easily graph such quantons' one-variable wavefunctions $\psi(x)$ and analyze them using single-variable calculus. Unless explicitly stated otherwise, we will *assume* this limitation in what follows.

For reasons beyond our scope here, quantum wavefunctions are also necessarily *complex* functions (meaning that $\psi(x)$ at a given x is a complex number) in many situations. In this text, though, we will confine ourselves to situations where complex numbers are not necessary.

Q9a.2 Wavefunctions and Position Probability

Our model states that the probability of locating a quanton at a given position is proportional to the square of the quanton's wavefunction at that position. But when we try to describe this mathematically, we immediately encounter a problem. Even in one dimension, there are an infinite number of possible positions, so the probability of being found at any specific mathematical position must be essentially zero!

The trick for dealing with this problem is to consider not a single position but a tiny *range* of positions between x and $x + dx$, where dx is so small that the value of $\psi(x)$ does not change significantly over that range. All of the (infinite number) of positions in this range have essentially the same probability, and collectively they add up to something small but finite. The collective probability of points in the range will also be proportional to the size of the range: doubling the range will double the number of included points, and since the points all have nearly the same probability, this doubles the collective probability. Therefore, the best way to state mathematically that position probability depends on the square of the wavefunction is as follows:

$$\Pr(x \text{ to } x+dx) \propto |\psi(x)|^2 dx \tag{Q9a.1}$$

where $\Pr(x \text{ to } x+dx)$ is a compact way of saying "The probability that the quanton's position will be somewhere in the range x to $x + dx$." I have written $|\psi(x)|^2$ instead of $[\psi(x)]^2$ or $\psi^2(x)$ because $|\psi(x)|^2$ is the correct notation when $\psi(x)$ is complex: it directs us to take the so-called "absolute square" of the complex number, which is always real and positive (which is crucial if the result is to describe a probability). The wavefunctions in this text will almost universally be real, so this is not an issue, but I'd like you to be familiar with the correct notation. Also remember that equation Q9a.1 only applies if dx is small enough that $\psi(x)$ is essentially constant over the range x to $x + dx$.

We can calculate the probability of finding the quanton within a larger range x_1 to x_2 by adding up the probabilities for a complete set of tiny ranges of infinitesimal width dx that span this larger range. Mathematically, we do this by evaluating an integral:

$$\Pr(x_1 \text{ to } x_2) \propto \int_{x_1}^{x_2} |\psi(x)|^2 dx \quad \text{or} \quad \Pr(x_1 \text{ to } x_2) = K\int_{x_1}^{x_2} |\psi(x)|^2 dx \tag{Q9a.2}$$

We can now fix the constant of proportionality K by noting that the probability of determining the quanton to be located between $-\infty$ and $+\infty$ (that is, *somewhere*) must be 1. Therefore, we must have

$$1 = \Pr(-\infty \text{ to } +\infty) = K\int_{-\infty}^{+\infty} |\psi(x)|^2 dx \quad \Rightarrow \quad K = \frac{1}{\int_{-\infty}^{+\infty} |\psi(x)|^2 dx} \tag{Q9b.3}$$

Therefore, quite generally, we have

The probability of finding a quanton's position to be within a certain range

$$\Pr(x_1 \text{ to } x_2) = \frac{\int_{x_1}^{x_2} |\psi(x)|^2 dx}{\int_{-\infty}^{+\infty} |\psi(x)|^2 dx} \tag{Q9a.4}$$

- **Purpose:** This equation expresses the link between a quanton's wavefunction $\psi(x)$ and the probability that quanton's position will be determined to be within the range x_1 to x_2.
- **Limitations:** This equation applies only to quantons confined to move in one dimension.

By doing the following exercise, you can convince yourself that we can multiply any wavefunction by overall constant A without changing the validity of equation Q9a.4.

Exercise Q9aX.1

Show that if we replace $\psi(x)$ in equation Q9a.4 by $\psi'(x) \equiv A\psi(x)$, we must get the same numerical result.

Indeed, we will find generally that only a wavefunction's *shape*, which describes the *relative* position probabilities of various regions, is physically relevant: we can define a wavefunction's *scale* to be anything we like (by multiplying it by any overall constant we like). However, it is both conventional and convenient to *define* a quantum wavefunction's scale so that

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 dx \equiv 1 \qquad \text{(Q9a.5)}$$

If we do this, we say that the quantum wavefunction is **normalized**, and equation Q9a.4 simplifies to being simply

$$\Pr(x_1 \text{ to } x_2) = \int_{x_1}^{x_2} |\psi(x)|^2 dx \quad \text{when} \quad \int_{-\infty}^{+\infty} |\psi(x)|^2 dx \equiv 1 \qquad \text{(Q9a.6)}$$

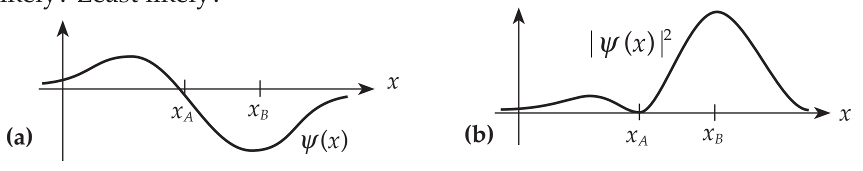
The simplicity of the left equation makes it handy to habitually work with *normalized* wavefunctions.

It is also crucial to note that for either equation Q9a.4 or equation Q9a.6 to work, $\int_{-\infty}^{+\infty} |\psi(x)|^2 dx$ must be finite and nonzero. This means that not all mathematical functions can be quantum wavefunctions. For example, the function $f(x) = Ax$ cannot be a valid quantum wavefunction, because the integral of $|f(x)|^2 = A^2x^2$ from $-\infty$ to $+\infty$ is $\frac{1}{3}A^2(\infty)^3 - \frac{1}{3}A^2(-\infty)^3 = \infty$. If a function's square integrated from $-\infty$ and $+\infty$ is a finite nonzero number, we say that the function is **normalizable** (because we can then multiply that function by a constant so that equation Q9a.5 is satisfied). Only *normalizable* functions can serve as quantum wavefunctions. In general, normalizable functions will go to zero as $|x|$ becomes large.

The following examples illustrate how to interpret the probability information implicit in a quantum wavefunction.

Example Q9a.1

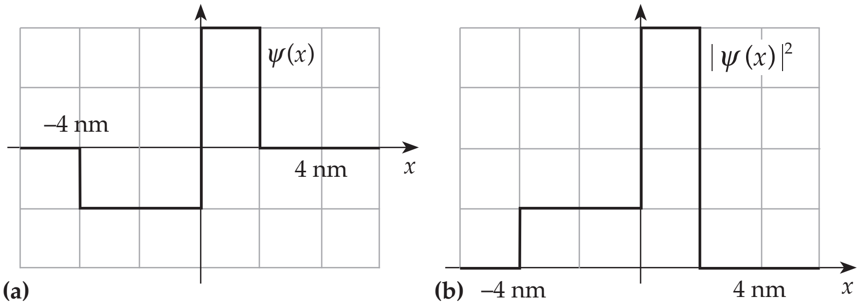
Problem: Suppose a quanton has the statefunction $\psi(x)$ shown to the left below. If we do an experiment to locate the quanton, what result is most likely? Least likely?



Solution The probability of getting a given x_i is $\propto |\psi(x_i)|^2$. The right-hand graph shows qualitatively what $|\psi(x)|^2$ looks like. As the probability is greatest where $|\psi(x)|^2$ is largest, the most likely result is position x_B [even though $\psi(x_B)$ is actually negative!]. The quanton is least likely to be found where the wavefunction crosses the x axis (at x_A) as $\psi(x_A) = 0 \;\Rightarrow\; |\psi(x_A)|^2 = 0$ there.

Example Q9a.2

Problem: Imagine that at a certain time, a quanton's wavefunction is as shown to the left below (a real wavefunction will never be this "boxy," but let's pretend). If we do an experiment that locates the quanton at this time, what is the probability that the result will be to the right of the origin?



Solution The graph to the right above shows the square of the wavefunction. It doesn't matter what the units of the vertical axis are: if the height of $\psi(x)$ is one unit, then the height of $|\psi(x)|^2$ is one unit squared; if the height of $\psi(x)$ is two units, the height of $|\psi(x)|^2$ is four units squared, and so on. Recall that a function's integral yields the area under the function's curve. We can see that there are four units squared of area under the graph of $|\psi(x)|^2$ to the right of $x = 0$ and six units squared under the whole graph (assuming that $\psi(x) = 0$ for points not in the diagram). So according to equation Q9a.4, the probability that the position will turn out to be to the right of zero is $4/6$ or $2/3$.

Exercise Q9aX.2

What is the probability that a position-determining experiment yields a result within ± 1 nm of the origin for the wavefunction in this example?

Example Q9a.3

Problem: Imagine that at a certain time, a quanton's wavefunction is

$$\psi(x) = A\sqrt{\frac{1}{x^2+b^2}} \tag{Q9a.7}$$

where A is a scaling constant and b is a constant with units of length. What is the probability of determining the quanton to be between $-b$ and b?

Solution We need to calculate the integral of $|\psi(x)|^2$. I would not expect you to be able to integrate this function from scratch, but one can easily look the indefinite integral up in a printed or online table of integrals. My table of integrals tells me that

$$\int |\psi(x)|^2 dx = A^2 \int \frac{1}{x^2+b^2} dx = A^2 \frac{1}{b} \tan^{-1}\left(\frac{x}{b}\right) \tag{Q9a.8}$$

According to equation Q9a.4, the probability is then

$$\Pr\begin{pmatrix} x \text{ between} \\ -b \text{ and } b \end{pmatrix} = \frac{\int_{-b}^{b} |\psi(x)|^2 dx}{\int_{-\infty}^{\infty} |\psi(x)|^2 dx} = \frac{\cancel{\frac{A^2}{b}}\left[\tan^{-1}\left(\frac{x}{b}\right)\right]_{-b}^{b}}{\cancel{\frac{A^2}{b}}\left[\tan^{-1}\left(\frac{x}{b}\right)\right]_{-\infty}^{\infty}}$$

$$= \frac{\tan^{-1}(1) - \tan^{-1}(-1)}{\tan^{-1}(\infty) - \tan^{-1}(-\infty)} = \frac{\frac{1}{4}\pi - (-\frac{1}{4}\pi)}{\frac{1}{2}\pi - (-\frac{1}{2}\pi)} = \frac{\frac{1}{2}\pi}{\pi} = \frac{1}{2} \quad \text{(Q9a.9)}$$

Note that this probability is unitless and less than 1, as it must be.

Exercise Q9aX.3

Sketch the graph of $|\psi(x)|^2 = A^2/(x^2 + b^2)$. Note that this function has a peak value of A^2/b^2 at $x = 0$, and goes to zero as x becomes large. Does it look plausible that half the area under the curve is between $x = \pm b$?

I have included this last example mostly for the sake of completeness. In this text, we will very rarely encounter wavefunctions where we have to do actual integrals to calculate probabilities. For our present purposes, it is enough to know that if we do an experiment to locate a quanton, the probability of finding it to be at a certain value x is proportional to $|\psi(x)|^2$, and in specific, the probability of finding a quanton's position to be within a certain *range* of positions is equal to the ratio of area under the curve of $|\psi(x)|^2$ within that range to same for *all* x. Note again that this is only physically meaningful if the area under the curve of $|\psi(x)|^2$ for all x is finite.

Q9a.3 Eigenfunctions and Eigenvalues

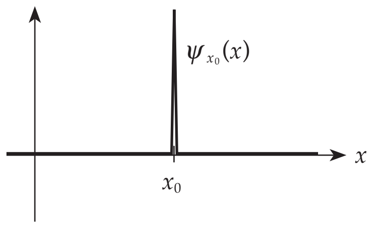


Figure Q9a.2
A position eigenfunction.

We have discussed how a quanton generally does not have a well-defined position. But we can define a normalizable quantum wavefunction whose quanton *does* have a given well-defined x-position x_0. Such a wavefunction would be a thin spike that is zero everywhere except within an infinitesimal range around x_0 (see figure Q9a.2). If we had a set of quantons all having this wavefunction and measure their positions, we would find *every single one of them* to have position x_0. Quantons described by this wavefunction therefore *do* have a well-defined position x_0.

If quantons in a given quantum state are *certain* to be measured to have a specific value a_0 for some measurable quantity A, we say that the quantum state is an **eigenfunction** for the **observable** A corresponding to the eigenvalue a_0. Our spike at position x_0 is therefore an eigenfunction for the position observable x corresponding to the particular position value x_0. ("Eigen" is a German word meaning "particular" or "idiosyncratic," so the sense is that an eigenfunction is uniquely associated with the particular observable value.) A spike located at position x_1 would be position eigenfunction for that eigenvalue. We see that for the position observable, we have an infinite set of position eigenfunctions, each corresponding to a spike at a particular position.

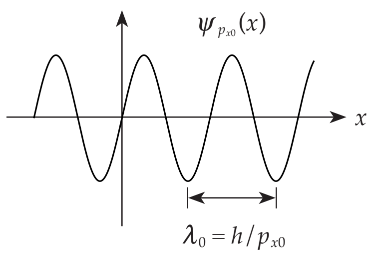


Figure Q9a.3
A x-momentum eigenfunction.

We can define eigenfunctions for other quantities as well. From the de Broglie relation, we know that a quanton's *momentum* is associated with the *wavelength* of its wavefunction: $|\vec{p}| = h/\lambda$. This means that an eigenfunction for the observable "momentum" must be a wavefunction with a precisely defined wavelength. In a one-dimensional world, such a wave would be a pure sinusoidal wave with constant amplitude (see figure Q9a.3). If the wavelength of this wave is precisely some value λ_0, then the corresponding x-momentum eigenvalue $p_{x0} = h/\lambda_0$. Again, we can define an infinite set of such sine waves, each corresponding to a specific momentum eigenvalue.

In addition to position and momentum, a quanton's "observables" include its kinetic energy K, potential energy V, total energy E, orbital angular momentum L around a given axis, and so on. For every observable, we will have an set of eigenfunctions, one eigenfunction for each value of that observable that we can possibly measure. If a quanton's wavefunction is one of these eigenfunctions, and we measure the corresponding observable, we will get the eigenvalue associated with that eigenfunction with certainty, meaning that quantons in that state have well-defined values for that observable. For a number of reasons, in future chapters we will be especially interested in the eigenfunctions associated with a quanton's total energy E, partly because this quantity is conserved in time for a quanton in an isolated system in the way that its position and momentum are not.

Note that physicists do *not* conventionally classify properties such as a quanton's mass m or charge q as "observables" even though they are measurable quantities, because for a given type of quanton (such as an electron), measuring such a quantity yields a certain value with certainty *no matter what the quanton's wavefunction might be*. We take a quantum "observable" to be a quantity (like position or momentum) whose results might vary for a given quanton and thus actually depend on the quanton's state.

Q9a.4 The Heisenberg Uncertainty Principle

Note also that technically we cannot *really* create a eigenfunction that corresponds to a mathematically precise position. Such a function would be a discontinuous spike of zero width that could not be integrated and thus normalized. We can imagine a spike whose width is so small that its width is of no practical consequence in a given situation, but we cannot really take that width to zero. This means that no matter how excellent our measuring devices might be, a set of quantons sharing the same spike-like wavefunction will be measured to have positions within some range $x \pm \Delta x$ where Δx may be very small but not zero. The uncertainty in a quanton's position for such a spike will therefore be approximately Δx.

Similarly, we cannot really create an eigenfunction corresponding to a mathematically precise momentum. In order for a wave to have an infinitely precise wavelength, we would need an infinitely long sine wave, and such a wavefunction cannot be normalized.

Why do we need an infinitely long sine wave? Here is a practical argument. Suppose that we can measure the distance between two adjacent crests of a sine wave to some accuracy $\pm\delta\lambda$ with our ruler. If we assume that the wave's wavelength is uniform, we could reduce the uncertainty in the measured wavelength by measuring the distance between, say, 10 crests and then dividing the result by 10: the uncertainty in the wavelength value is then only $\delta\lambda/10$. By including more and more crests in our measurement, we can reduce the uncertainty further, but we cannot reduce it to zero unless we include an infinite number of crests.

More mathematically, the Fourier theorem states that a waveform of any shape can be constructed by adding sine waves with appropriate wavelengths, phases, and amplitudes. Consider a wave that looks like a sine wave but goes to zero outside a range of some length L (see figure Q9a.4). A pure sine function by definition oscillates with a constant amplitude from $-\infty$ to $+\infty$, so what we have is not a pure sine function. Therefore it must be a *sum* of pure sine functions. Indeed, what we want will be the sum of sine waves of slightly different wavelengths that constructively interfere where the wave amplitude is large in the middle of the wave, but get out of phase and cancel

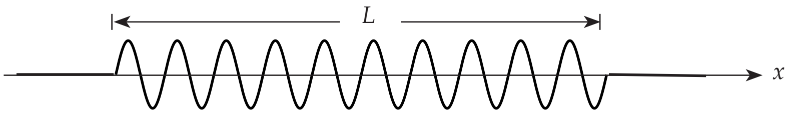


Figure Q9a.4
A sine-like wavefunction that is nonzero only for a span of width L.

each other out when we are outside the spatial range of length L. This means that our space-limited sine wave is not a pure sine wave at all, but rather a sum of sine waves with a small but nonzero range of wavelengths. Since it is not a pure sine wave, it is not a pure momentum eigenfunction, meaning if we measure the momenta of a set of quantons with this wavefunction, we will obtain not a single well-defined momentum value, but rather a range of momenta corresponding to the range of sine-wave wavelengths that went into the sum.

Note that eigenfunctions of position and those for momentum are incompatible. If we have a set of quantons that all have a wavefunction that is a very long sine wave and we measure their momenta of half of them, we will get results in a vary narrow range centered on the momentum corresponding to the nearly infinite sine-wave's wavelength: the quantons' momentum is well defined. But if we measure the positions of the other half, we will get values ranging almost from $-\infty$ to $+\infty$: their position is poorly defined. Conversely, if a set of quantons have a common wavefunction that allows their position to be well-defined (the wavefunction is spike-like), then their momentum will be poorly defined (the wavefunction looks nothing like an infinite sine wave).

Indeed, we can put some quantitative limits on the uncertainties involved. Consider the space-limited sinusoidal wave shown in Figure Q9a.4. If we measure the positions of quantons having this wavefunction, we will get results within $\pm\frac{1}{2}L$ of the center of the wavefunction, so the uncertainty in the quantons' position is about $\Delta x \approx \frac{1}{2}L$. If the wavelength of the sine wave inside this range is λ, then we have $N = L/\lambda$, so our practical uncertainty in the wavelength is roughly $\Delta\lambda = \delta\lambda/N = \lambda\,\delta\lambda/L$. If this uncertainty is small, we can use calculus to estimate the uncertainty in the momentum:

$$\Delta p_x \approx \left|\frac{dp_x}{d\lambda}\right|\Delta\lambda = \left|\frac{d}{d\lambda}\left(\frac{h}{\lambda}\right)\right|\Delta\lambda = \frac{h}{\lambda^2}\Delta\lambda \qquad \text{(Q9a.10)}$$

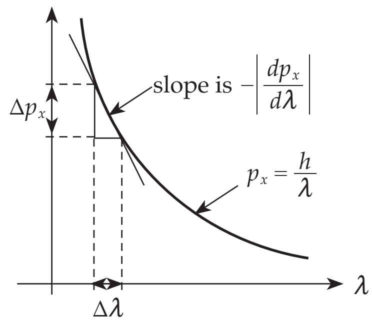


Figure Q9a.5
If measured wavelengths are in the range $\Delta\lambda$, then the corresponding x-momenta will be in the range Δp_x. (The slope's negative sign only tells us that increasing λ decreases p_x. We are interested only in the *magnitudes* of the connected ranges, so we take the slope's absolute value.)

This equation basically says that if we wiggle the value of the wavelength λ within a range $\Delta\lambda$, the value of the momentum p_x wiggles by Δp_x given by the expression above (see figure Q9a.5). Now assume that the basic measurement uncertainty in the wavelength $\delta\lambda$ is some small fraction α of the wavelength (because determining *exactly* where a crest is will be harder for longer than shorter wavelengths, probably in direct proportion to the wavelength), so $\delta\lambda \approx \alpha\lambda$, where α is a small unitless number. Substituting this into the relation $\Delta\lambda = \delta\lambda/N = \lambda\,\delta\lambda/L$ and the result into equation Q9a.10 yields

$$\Delta p_x \approx \frac{h}{\lambda^2}\frac{\lambda\alpha\lambda}{L} = \frac{h\alpha}{L} \quad \Rightarrow \quad \Delta p_x \Delta x \approx \Delta p_x\left(\tfrac{1}{2}L\right) \approx \frac{h\alpha}{2} \qquad \text{(Q9a.11)}$$

We see that the product of the position and momentum uncertainties for a wavefunction of the type shown in Figure Q9a.4 is some unitless number times Planck's constant h. This tells us a very interesting thing: if we have a set of quantons described by same quantum wavefunction, and we measure the x-positions of half and the x-momenta of the other half, the uncertainties of these measurements are in fact *inversely proportional*: the more certain the wavefunction implies the quantons' positions are, the less certain their momenta are and vice versa, with the constant of (inverse) proportionality being some smallish unitless multiple of Planck's constant h.

An exact mathematical calculation taking into account the exact spread of wavelengths that the Fourier theorem implies for our space-limited wavefunction and the precise mathematical definition of the uncertainties (defined to be the standard deviations of the measurement results that we obtain for a theoretically infinite set of quantons) yields the same result with $\alpha = 1/2\pi$:

The quantitative statement of the Heisenberg Uncertainty Principle

$$\Delta p_x \, \Delta x \geq \frac{h}{4\pi} = \frac{\hbar}{2} \quad \text{where } \hbar \equiv \frac{h}{2\pi} \tag{Q9a.12}$$

The proof also establishes that $\hbar/2$ is the *lower limit* of this product: not all wavefunctions are as good as our space-limited sine wave.

This equation is the quantitative statement of the famous **Heisenberg Uncertainty Principle.** Qualitatively this principle states that

> No quantum wavefunction gives its quantons both well-defined positions and well-defined momenta.

The HUP follows because position and momentum eigenfunctions are incompatible

We have seen that the qualitative Heisenberg Uncertainty Principle (HUP) is a simple consequence of the fact that momentum eigenfunctions are not the same as position eigenfunctions. A quanton's statefunction simply cannot simultaneously be a spike at a well-defined position *and* an infinite sinusoidal wave with a well-defined wavelength!

Caveats about the meaning of the HUP

Because of the mystique that the Heisenberg Uncertainty Principle has gained in the popular press, people sometimes treat it as if it were a fundamental principle of quantum mechanics. I hope you can see from the discussion above that it is not fundamental: it is a necessary *consequence* of the concept of the quantum wavefunction.

People also sometimes use equation Q9a.12 to "derive" various quantum-mechanical results. However, such derivations are often conceptually dubious (at best), and might more honestly be described as calculations based on simple dimensional analysis (that is, simply making sure that the units are right). Keep in mind that the quantitative Heisenberg Uncertainty Principle technically expresses a relationship between the standard deviations of results obtained from experiments to measure x-position and x-momentum p_x performed on two halves of a very large set of quantons all having the same initial wavefunction $\psi(x)$.

Thus the principle does not really tell us anything useful about the results we would obtain when performing these measurements on a *single* quanton. Indeed, we will see in chapter Q9b that trying to perform such measurements simultaneously on a single quanton is impossible by definition, and even performing such measurements sequentially does not tell us what we might think. So be suspicious of any argument that seeks to apply the principle to a single quanton. The principle more properly describes characteristics or implications of a given quantum *wavefunction*, characteristics that can only be quantified by doing experiments on a large number of quantons that all are described by that given wavefunction.

However, the Heisenberg Uncertainty Principle is valuable, partly because it underlines the demise of the classical picture of the universe. Not only are quantons not particles with classical trajectories, we cannot even define a quanton's classical state because it cannot simultaneously have a well-defined position $\vec{r}$ and momentum $\vec{p}$! We must decisively set aside the classical picture and instead embrace the quantum wavefunction as the new foundation for understanding how the universe works.

ANSWERS TO EXERCISES

Q9aX.1 Since $\int a\, f(x)\,dx = a\int f(x)\,dx$ for any function $f(x)$ and constant a, we have

$$\frac{\int_{x_1}^{x_2} |\psi'(x)|^2 dx}{\int_{-\infty}^{+\infty} |\psi'(x)|^2 dx} = \frac{\int_{x_1}^{x_2} |A\psi(x)|^2 dx}{\int_{-\infty}^{+\infty} |A\psi(x)|^2 dx} = \frac{\int_{x_1}^{x_2} A^2|\psi(x)|^2 dx}{\int_{-\infty}^{+\infty} A^2|\psi(x)|^2 dx}$$

$$= \frac{\cancel{A^2}\int_{x_1}^{x_2} |\psi(x)|^2 dx}{\cancel{A^2}\int_{-\infty}^{+\infty} |\psi(x)|^2 dx} = \frac{\int_{x_1}^{x_2} |\psi(x)|^2 dx}{\int_{-\infty}^{+\infty} |\psi(x)|^2 dx} \qquad \text{(Q9a.13)}$$

Q9aX.2 Note that the total area under the curve of $|\psi(x)|^2$ is six boxes, while the total area under $|\psi(x)|^2$ between $x = -1$ nm and $x = 1$ nm is four half-boxes (to the right) + 1 half-box (to the left) = $2\frac{1}{2}$ boxes. Therefore the probability is $2\frac{1}{2}/6 = 5/12$.

Q9aX.3 A graph of the function appears in figure Q9a.6. (You can make a reasonable sketch by plotting values at $x = \pm b$, $\pm 2b$ and $\pm 3b$ and connecting with a smooth curve.) The shaded area is indeed plausibly half the total.

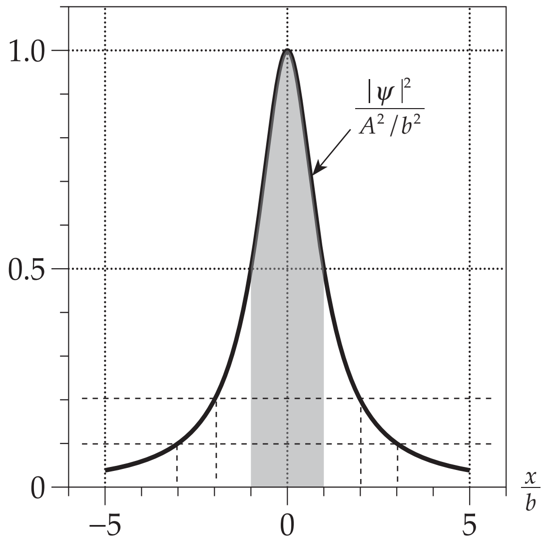


Figure Q9a.6
A graph of $|\psi|^2 = A^2/(x^2+b^2)$. The gray region shows the area under the curve between $x = \pm b$.

Q9b The Collapse of the Wavefunction

Chapter Overview

Introduction
In chapter Q9a, we saw that using a wavefunction to describe a quanton's state could explain most of what we observe in a quanton-at-a-time two-slit interference experiment. But we have not yet addressed the fact that when we attempt to observe which slit the quanton goes through using proximity detectors, we destroy the interference pattern. In this chapter, we will discuss how we must modify the model to account for this result, and the strange and disturbing implications of those modifications.

Section Q9b.1: Observation Changes the Observed
In order to explain what we observe in a quanton-at-a-time two-slit interference where we place proximity detectors at the two slits, we must assume that process of measuring the quanton's position must actively change the quanton's wavefunction so that it becomes a position eigenfunction (a spike) corresponding to the position value observed. As discussed in the section, this nicely explains why deploying proximity detectors ruin the interference pattern.

Section Q9b.2: This Implies Consistency
This active modification of the wavefunction also implies that immediately successive position measurements will yield consistent (instead of random) results. Even in the absence of experimental evidence, we might conclude that such consistency is *essential* for a "measurement" to have any meaning at all, and therefore that the process of measuring of *any* quantum observable must cause a quanton's wavefunction to become the eigenfunction that corresponds to the value of the observable that was measured. We will take this as a fundamental feature of our quantum model.

Section Q9b.3: Calculating Probabilities
As in the case of position, we can generally only predict the result of measuring an observable probabilistically. The probability of a given result plausibly depends on how "similar" the original wavefunction is to the observable eigenfunction that it must become. The section argues that the following expression allows us to compute this probability:

$$\Pr(\psi \to \phi) = \left| \int_{-\infty}^{+\infty} \psi^*(x)\phi(x)\,dx \right|^2 \qquad \text{(Q9b.7)}$$

- **Purpose:** This equation specifies the probability that if we perform a measurement on a quanton whose wavefunction is $\psi(x)$ and end up with a result whose eigenfunction is $\phi(x)$. The quanton's wavefunction following the measurement is $\phi(x)$.
- **Limitations:** This applies to quantons confined to move in one dimension. $\psi(x)$ and $\phi(x)$ may be complex but must be normalized.
- **Note:** For real functions, $\psi^*(x) = \psi(x)$.

Section Q9b.4: Collapsing Wavefunctions

The idea that measurement forces a change in a quanton's wavefunction presents thorny theoretical difficulties, some that remain unresolved to the present. Physicists describe this phenomenon as the "collapse of the wavefunction," because if (for example) we measure a quanton's position, its wavefunction becomes a position eigenfunction (a spike) and the value of the wavefunction everywhere else must instantly collapse to zero. Normal waves don't behave like this, and the time-dependent Schrödinger equation (which nicely describes the time evolution of quantum wavefunctions in every other context) does not predict this behavior. This collapse also seems to violate special relativity's requirement that effects cannot travel faster than light, and its irreversible character seems to contradict the principle that basic equations of mechanics are time-reversible.

Section Q9b.5: The EPR Argument and Bell's Theorem

Two famous thought experiments helped expose the theoretical problems that the quantum model presents and clarify physicists' thinking about those problems. One of these was the EPR argument, proposed by Einstein, Podolsky, and Rosen in 1935. This thought experiment showed that an experiment at point *A* could (through the collapse phenomenon) affect the results observed in an experiment performed at a distant point *B*, even if the experiments were performed so close together in time that a signal from *A* could not reach point *B* even at the speed of light. Since superluminal connections are not allowed by relativity, Einstein believed that this thought experiment demonstrated that quantum mechanics was wrong (or at best incomplete).

Until the 1960s, most physicists thought that Einstein's argument was merely a metaphysical argument without empirical consequences. But in 1964, John Bell showed mathematically that any theory that was local (that is, allowing no superluminal communication between distant points) and involved real properties (properties that have objective values independent of measurement) would lead to experimental results that could be distinguished from the predictions of quantum mechanics. In the early 1980s, Alain Aspect and collaborators were able to test this prediction, and their results supported quantum model and thus (by Bell's theorem) excluded any possible local and real theory of the type that Einstein supported.

Theoretical work also showed that the superluminal connections implied by quantum mechanics did not actually violate relativity: the probabilistic nature of quantum mechanics "hides" the superluminal connections in such a way that they cannot be used to send a signal faster than light. The section discusses in more detail how this works.

Section Q9b.6: Schrödinger's Cat

Also in 1935, Erwin Schrödinger proposed a thought experiment that exposed problems with the superposition principle when conjoined with the collapse of the wavefunction. His thought experiment imagined putting (the equivalent of) a chain of objects in a closed box that connected the life or death of a cat with the quantum state of an electron that we can consider to be the superposition of a set of position eigenfunctions, some of which lead to the cat's death. If we can consider the quantum model to apply to the entire system, then before we "measure" the system, the cat is in a superposition of "dead" and "alive" quantum states. But although we can and do observe microscopic systems in wavefunctions that correspond to superpositions of various observables, we never observe cats to be in a superposition of "dead" and "alive" states. But if cats are quantum systems just like electrons are, why not? If the collapse of the quantum system's state occurs earlier along the chain of devices than the cat, where does it occur and why?

The section discusses experimental evidence regarding how large an object can be and still be observed to be in a superposition state, as well as a number of possible solutions to the theoretical of where the collapse occurs (some quite bizarre), none of which has gained the consensus of the community as a whole.

Q9b.1 Observation Changes the Observed

We have not yet accounted for the fact that proximity detectors spoil the pattern

In chapter Q9a, we saw that using a wavefunction to describe a quanton's state could explain most of what we observe in a quanton-at-a-time two-slit interference experiment. But we have not yet addressed a crucial observation: when we attempt to observe which slit the quanton goes through by installing proximity detectors at each slit, we *destroy* the interference pattern, no matter how cleverly we design those detectors. Our model does not yet provide any explanation of this curious fact.

The Möbius strip analogy presented in the last chapter suggests some guidance here. Recall that to give the strip's half-twist a well-defined location, we had to actively deform the strip to make the rest of the strip flat. The result was indeed a well-defined location for the twist, but at the cost of modifying the state of the Möbius strip. It doesn't matter what means or device we use flatten the rest of the strip (our hands, an iron, or a wringer), because the abstract process of locating the twist itself *requires* flattening the rest of the strip: there is no other way to confine the twist.

The act of measurement must *change* the quanton's quantum wavefunction

We can explain the proximity-detector result if we assume that a position measurement does something similar. We will assume that conceivable process for locating a quanton must first give the quanton a location to report by changing its wavefunction to one that *has* a well-defined position, that is, a position eigenfunction (a spike). The probability that this modification yields any particular position result is proportional to the square of the original wavefunction at that position (as we have seen), but the additional feature is that the process necessarily forces the wavefunction into the position eigenfunction corresponding to the observed result. *The act of observation changes the quantum state of what is observed.*

This nicely explains what installing proximity detectors does. By placing a proximity detector at each slit, we compel the quanton's original wavefunction to develop a location (one slit or the other), and since the value of the original wavefunction was the same at each slit, we have a probability of 0.5 for the quanton to be found at each slit. But once the result has been registered, the quanton's wavefunction is now a spike located at the corresponding slit. This quanton's wavefunction will then evolve exactly as if it were a pulse wave emerging from *that particular single slit*, not simultaneous waves emerging from the two slits, and the probability pattern that gets displayed on this screen will be a single-slit diffraction pattern, not a double-slit interference pattern. This is exactly what we observe.

Q9b.2 This Implies Consistency

This assumption is necessary for successive measurements to be consistent

This assumption has another highly desirable implication. Suppose that I put a proximity detector A at slit A and a proximity detector B at slit B. Suppose then that I put a third proximity detector C immediately following detector B in the direction that the wave moves (see figure Q9b.1). Suppose that for a given quanton, detector A fires (indicating that it has located the quanton at slit A). Since the quanton's wavefunction immediately after this detection is a spike located at detector A, the wavefunction at detector C is now zero, and the probability that detector C fires is therefore zero. On the other hand, if detector B fires, then the quanton's wavefunction immediately afterward is a spike at slit B, and this spike wave will move through the immediately following detector C with certainty, meaning that if B fires, then C will certainly fire as well. This means that the position measurements are *consistent*: once

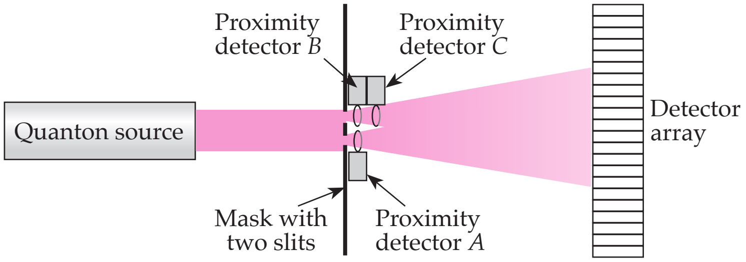


Figure Q9b.1
Experiment with a third proximity detector C. For consistency detector C must fire if and only if detector B fires.

we have located a quanton, if we immediately measure its position again, we get the same result *with certainty*.

Consider now the alternative. Suppose the wavefunction did *not* become a position eigenfunction. Since the original wavefunction had a nonzero value at detector C, that detector would have a nonzero (but also non-certain) probability of firing no matter whether it was detector A or B that fired first. So I could easily have the situation that detector C fires just after detector A, or that detector C does *not* fire immediately after detector B fires. So where is the quanton? In this situation it would seem like the quanton is ricocheting around like the ball in a pinball machine. We would conclude that a measurement of the quanton's position *has no meaning*: the detectors might as well be firing completely randomly.

This requirement of measurement consistency is indeed so important that we might have decided *on that basis alone* that, given the wavefunction model, a measurement logically *must* modify a quanton's wavefunction in the manner we have described. (Fortunately, our observations in the two-slit experiment do not falsify this prediction, strengthening our confidence in the model we are developing.)

In general: *measurement forces the quanton's state to become the eigenfunction corresponding to the result obtained*

This consistency principle implies that what applies to position measurements must in fact also apply measurement of any quantity. If we measure a quanton's x-momentum to be some value p_{x0}, then an immediate subsequent measurement of x-momentum should certainly yield the same result, which will only happen if the quanton's wavefunction after the first measurement is now the momentum eigenfunction corresponding to the eigenvalue p_{x0}. In general, we must assume that measuring an observable A converts a quanton's wavefunction to the eigenfunction whose eigenvalue is the result a_0 obtained by the measurement.

Q9b.3 Calculating Probabilities

How can we calculate the probability of this change

Of course (as in the case of position), *which* of the possible results we get (and thus which eigenfunction the wavefunction subsequently becomes) is generally only determined probabilistically, with the probability depending on something about how similar the original wavefunction is to the result's eigenfunction. Certainly if the original wavefunction is the *same* as a certain result's eigenfunction, then we should get that result with certainty (probability 1). But how can we quantify how "similar" wavefunctions might be?

A hypothesis to test

Suppose we have two real and *normalized* wavefunctions $\psi(x)$ and $\phi(x)$. Consider the "similarity" number s defined by the expression

$$s \equiv \int_{-\infty}^{+\infty} \psi(x)\phi(x)\,dx \tag{Q9b.1}$$

Let's see whether quantity has the properties that we want for a probability.

Suppose that the two functions are actually the *same* function $\psi(x)$. Then the expression becomes

$$s \equiv \int_{-\infty}^{+\infty} |\psi(x)|^2 dx = 1 \qquad \text{(Q9b.2)}$$

(the last result follows because we are assuming the wavefunction is normalized). This is consistent with what we want, because if the wavefunction becomes a certain observable's eigenfunction after measurement and we measure the same observable again immediately, then we want the probability of getting the same result (with the same final eigenfunction) to be 1.

Conversely, suppose that a measurement causes the quanton's wavefunction to become a certain eigenfunction. We want the probability of a result corresponding to a *different* eigenfunction (of the same observable) to be zero, because if we measure the same observable again immediately, we must get the same result with certainty, meaning that the probability of all *other* results must be zero. To be concrete, suppose that we are considering two position eigenfunctions $\psi_{x_0}(x)$ and $\psi_{x_1}(x)$ that are infinitesimally thin spikes at x-positions x_0 and x_1 respectively. Because these functions are nonoverlapping spikes, wherever one is nonzero, the other will be zero. This means their product is zero at all values of x, implying that (as desired)

$$s \equiv \int_{-\infty}^{+\infty} \psi_{x_0}(x)\,\psi_{x_1}(x)\,dx = 0 \qquad \text{(Q9b.3)}$$

The only problem with our similarity value s is that it could be negative, for example, if $\phi(x) = -\psi(x)$. This would be bad for a probability. But we can fix this by defining the probability to be the *square* of s. This would not ruin the results above, because 1^2 is still 1 and 0^2 is still 0. Our final proposed expression for the probability of starting with a normalized wavefunction $\psi(x)$ and ending up after a measurement with the quanton's state being a normalized eigenfunction $\phi(x)$ is therefore

$$\Pr(\psi \to \phi) = s^2 \equiv \left[\int_{-\infty}^{+\infty} \psi(x)\phi(x)\,dx\right]^2 \qquad \text{(Q9b.4)}$$

Checking that the final hypothesis is consistent with how the wavefunction predicts position probabilities

Let's see if this proposal specifically works in the case where measuring the position of quanton with a normalized wavefunction $\psi(x)$ yields the result x_0. The probability of this result *should* be the square of the integral of $\psi(x)$ with $\psi_{x_0}(x)$, where the latter is a normalized spike at x_0. To make the integral possible, consider the spike to be a rectangular bar with a tiny width δx_0 and height $\sqrt{1/\delta x_0}$. The *square* of this function will be a rectangular bar with the same width δx_0 but a height $1/\delta x_0$. The area under the curve of the squared bar is thus $\delta x_0/\delta x_0 = 1$, as required if the original bar function is to be normalized. Now the product of $\psi(x)$ and the original unsquared bar is nonzero only between x_0 and $x_0 + \delta x_0$, so the product $\psi(x)\psi_{x_0}(x)$ is only nonzer in the same range, and our similarity integral becomes

$$\int_{-\infty}^{+\infty} \psi(x)\,\psi_{x_0}(x)\,dx = \int_{x_0}^{x_0+\delta x_0} \psi(x)\left(\frac{1}{\sqrt{\delta x_0}}\right)dx = \frac{1}{\sqrt{\delta x_0}}\int_{x_0}^{x_0+\delta x_0} \psi(x)\,dx \qquad \text{(Q9b.5)}$$

Now, if the spike's range δx_0 is sufficiently small, the value of $\psi(x)$ will be approximately constant with the value $\psi(x_0)$ over that range. We can pull the constant out of the integral, yielding

$$\int_{-\infty}^{+\infty} \psi(x)\,\psi_{x_0}(x)\,dx = \frac{\psi(x_0)}{\sqrt{\delta x_0}}\int_{x_0}^{x_0+\delta x_0} dx = \frac{\psi(x_0)}{\sqrt{\delta x_0}}\delta x_0 = \psi(x_0)\sqrt{\delta x_0} \qquad \text{(Q9b.6)}$$

The square of this is simply $|\psi(x_0)|^2\delta x_0$, which is exactly what we stated in the last chapter the probability *should* be for measuring position and getting a result between x_0 and $x_0 + \delta x_0$ when δx_0 is sufficiently small. This gives us some confidence that equation Q9b.4 indeed is the correct expression.

The correct expression for possibly complex wavefunctions is

$$\Pr(\psi \to \phi) = \left| \int_{-\infty}^{+\infty} \psi^*(x)\phi(x)\,dx \right|^2 \qquad \text{(Q9b.7)}$$

- **Purpose:** This equation specifies the probability that if we perform a measurement on a quanton whose wavefunction is $\psi(x)$ and end up with a result whose eigenfunction is $\phi(x)$. The quanton's wavefunction following the measurement is $\phi(x)$
- **Limitations:** This applies to quantons confined to move in one dimension. $\psi(x)$ and $\phi(x)$ may be complex but must be normalized.

The final result for normalized (but possibly complex) wavefunctions

The asterisk and the $|\ \ |^2$ notation mean something to people who understand complex numbers, when the wavefunctions are real (as they will be in this text except in chapter QA and a few isolated problems) then $\psi^*(x) = \psi(x)$ and the $|\ \ |^2$ notation is the same as taking the ordinary square.

Q9b.4 Collapsing Wavefunctions

The fact that a measurement forces the a quanton's quantum state to become the eigenfunction consistent with the observed result, though it seems almost required by consistency and bolstered by experiment, presents a host of thorny theoretical problems that remain unsolved almost a century after the concept was first proposed. We will briefly explore these problems in the remaining sections of this chapter.

Why our measurement rule is described as a "collapse"

Physicists call this phenomenon the "collapse of the wavefunction." One can understand this description's force by considering the measurement of a quanton's position. Before the measurement, the quanton's wavefunction $\psi(x)$ is a nice ordinary wave with nonzero values over a wide range of positions in space. Immediately after the measurement, however, the wavefunction is a spike located at only one position. The values of the quanton's wavefunction at *all* other points in space have instantaneously collapsed to zero!

Theoretical problems with this picture…

This is not like the behavior of any other kind of wave we know about. How can this even happen? How can what a tiny position detector is doing in a small region of space affect what is going on everywhere else?

(1) problems with relativity?

Moreover, special relativity makes it clear that the whole idea of something "instantaneously" affecting a large region of space is highly suspect. Not only can no causal effect travel faster than the speed of light (making instantaneous action-at-a-distance impossible), but what one observer considers "instantaneous" effect in a region of space may not look instantaneous at all to an observer in different reference frame.

(2) two time-evolution rules?

Wavefunction collapse also means that how wavefunction evolves in time is described by two completely distinct rules. When we are *not* doing any measuring, the wavefunction evolves smoothly, continuously, and deterministically (like classical waves evolve) according to the time-dependent Schrödinger equation (TDSE). During a measurement, however, the wavefunction changes discontinously and undeterministically (since we know only probabilistically the eigenfunction to which it evolves). No one knows how to make the TDSE produce behavior like this, and certain mathematical arguments seem to imply that the TDSE *cannot* do this (at least without modification). But having two distinct rules for describing the time-evolution of a wavefunction is troubling.

(3) Collapse is irreversible?

Part of the argument that the TDSE as it stands cannot describe this behavior is because the collapse process is *irreversible*. The laws of classical particle or wave mechanics and the TDSE are all are *time-reversible* in that if we replace time t by $-t$, the equation still works. For example, a video of two Newtonian particles colliding elastically would show a perfectly valid physical process even the video were reversed. But since many different original wavefunctions could all collapse to the same spike, we cannot reverse whatever is happening in the measurement process: information about the original shape of the wavefunction seems to be irretrievably lost.

In unit T, we will explore processes involving macroscopic objects (objects consisting of a huge number of elementary particles) that do indeed seem irreversible (such as ice melting on a warm day). We will see in that unit how such behavior can arise from time-reversible laws of mechanics. It is possible that because measurement devices are typically macroscopic, the irreversibility arises when a microscopic quantum system interacts with the macroscopic measuring device. But no one yet has been able to come up with an accepted mathematical description of this process, and if the measurement device itself is described by a quantum wavefunction obeying the TDSE, then there are good (but subtle) mathematical reasons to believe that genuine collapse of the entire system is still not possible.

Q9b.5 The EPR Argument and Bell's Theorem

During the past century, physicists have worked very hard to resolve these issues, and there has been some progress. In particular, two thought experiments have famously exposed the problems that the quantum model implies and helped clarify physicists' thinking about those problems.

Description of the EPR thought experiment

One of these thought experiments is the Einstein-Podolsky-Rosen argument, first proposed by Albert Einstein, Boris Podolsky, and Nathan Rosen in a paper published in 1935. Consider a pair of identical quantons produced by the decay of a quanton originally at rest, say perhaps a pair of photons produced by the decay of a π^0 meson. Suppose that photon A travels from this decay to a laboratory operated by Aisha, and the other (photon B) in the opposite direction to a laboratory operated by Beto. Because momentum is conserved, and the original meson was at rest, $\vec{p}_A + \vec{p}_B = 0 \;\Rightarrow\; \vec{p}_B = -\vec{p}_A$, where $\vec{p}_A$ and $\vec{p}_B$ are the momenta of photons A and B respectively. Moreover, if the original meson was at the origin, then at any given instant we also have $\vec{r}_B = -\vec{r}_A$ (where $\vec{r}_A$ and $\vec{r}_B$ are the photon's respective positions), because the photons travel in opposite directions at the same speed (the speed of light) for the same amount of time.

Suppose that Aisha measures photon A's position. Since we know that $\vec{r}_B = -\vec{r}_A$, if we know photon A's position, we also know photon B's position precisely, and Beto *must* get the result implied by this equation if Beto happens to measure B's position at that time. If Aisha happens to measure photon A's momentum, then because we must have $\vec{p}_B = -\vec{p}_A$, we also know photon B's momentum precisely and thus the result that Beto *must* get if he measures momentum as well.

If faster-than-light effects are not possible, then photon B must have a well-defined position and momentum

Here now is the crux of the EPR argument. Assume that (as relativity would seem to dictate) that what Aisha does in her lab cannot possibly affect what Beto observes in his, because even a signal traveling at the maximal speed of light from Aisha's lab could never catch up to photon B. So since Aisha could freely choose whether to measure position or momentum, and since we know precisely what Beto will observe in either case, and what Beto observes cannot be affected by what Aisha does, photon B must already have both a well-defined position *and* a well-defined momentum before Beto ever makes his measurement. This contradicts the whole wavefunction model we have constructed and in particular, the Heisenberg Uncertainty Principle that follows from that picture. This means that quantum mechanics is wrong.

Alternatively, if the quantum model is correct, we must have faster-than-light connections

On the other hand, suppose the quantum picture is correct. Then, when Aisha measures photon A's position, its wavefunction instantly collapses to a peak at position $\vec{r}_A$, and the requirement that $\vec{r}_B = -\vec{r}_A$ means that what Aisha does must simultaneously collapse photon B's wavefunction to a spike at $\vec{r}_B = -\vec{r}_A$ so that Beto gets the right result if he happens to measure position at the same time. Alternatively, if Aisha measures momentum, then the wavefunctions of both her photon and Beto's photon must simultaneously collapse to the momentum eigenfunctions corresponding to results $\vec{p}_A$ and $\vec{p}_B = -\vec{p}_A$, respectively. This preserves the quantum model but the expense of requiring that what Aisha does affect the quantum state of photon B superluminally (that is, over a distance larger than what light could travel during the duration of the experiment).

The argument makes it clear that we physics can either be "local" (effects cannot travel faster than the speed of light) OR we can have wavefunctions of the type described by the quantum model, not both. Einstein and his collaborators strongly believed in the principle of locality, and so presented this argument to claim that the wavefunction picture of quantum mechanics was wrong (or, at best, incomplete).

Bell shows that we can resolve this question experimentally, and the quantum model emerges triumphant

For about 30 years after the EPR paper was published, most physicists considered this argument to be more metaphysical than physical. Who could say what mechanisms were actually involved here, and quantum mechanics was spectacularly successful experimentally, so why worry about it? But in 1964, John Bell published a theorem that proved that these alternatives could in fact be distinguished experimentally. In early the 1980s, technology finally made it possible to perform experiments of the type envisioned by Bell, and Alain Aspect and his collaborators were able to offer conclusive experimental evidence supporting the quantum model, even in situations where this would require superluminal connections between the two photons' states.

Why this actually does *not* contradict relativity.

Physicists working on implications of the EPR argument also were eventually able to show that the superluminal connections did *not* materially contradict the theory of relativity. Considering Beto working alone in his laboratory. He arbitrarily measures the momenta of *some* of the photons he receives (getting results $\vec{p}_{B1}, \vec{p}_{B3}, \vec{p}_{B6}, \ldots$ for photons 1, 3, and 6, for example) and the positions of others (getting results $\vec{r}_{2B}, \vec{r}_{4B}, \vec{r}_{5B}, \ldots$ for photons 2, 4, and 5). But whatever Aisha chooses to measure at her end (in spite of the fact that her choices actively collapse the state of each of Beto's photons to various eigenfunctions) don't actually change what to Beto simply look like lists of random results. Since is no way that he can tell by looking at his lists what Aisha is doing, so she cannot send a superluminal signal to Beto by choosing to measure, say, position for photon 5. Only when they get together later (each traveling at less than the speed of light to a common location) will they see that whenever they *happened* to measure the same quantity (say, momentum for photon 3 and position for photon 5) they *always* observe that $\vec{p}_{3B} = -\vec{p}_{3A}$ and $\vec{r}_{5B} = -\vec{r}_{5A}$. So there is a genuine and physically real superluminal *correlation* between their results, but one that can only can be observed once the two lists have been transported at the speed of light or slower to a common location. The randomness inherent in quantum mechanics hides the effect of the correlation until the two lists are compared side-by-side. This allows quantum mechanics to sidestep the requirements of relativity.

Why physicists are still bothered

Even so, physicists remain troubled by the necessity of what Einstein called "*spukhafte Fernwirkungen*" ("spooky actions at a distance"). Richard

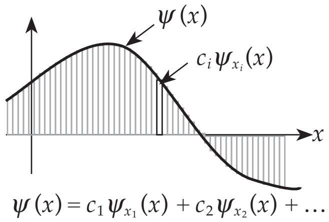


Figure Q9a.2
We can consider any wavefunction to be a sum of position eigenfunctions $\psi_{x_i}(x)$ (depicted here as thin bars) multiplied by appropriate coefficients c_i.

Feynman, one of the greatest theoretical physicists of the mid-20th century wrote the following in 1982: "We have always had a great deal of difficulty understanding the world view that quantum mechanics represents. At least I do… I cannot define the real problem, therefore I suspect that there's no real problem, but I am not sure there's no real problem."

Q9b.6 Schrödinger's Cat

The concept of superposition also presents issues when the collapse of the wavefunction is involved. Superposition means that one can, for example, consider a general wavefunction to be a *superposition of possible position outcomes*: we can consider any wavefunction to be a sum of position eigenfunction spikes multiplied by appropriate coefficients (see figure Q9b.2). Measuring position therefore amounts to making one of these possibilities real by selecting its eigenfunction and erasing all of the others.

Also in 1935, Erwin Schrödinger (author of the time-dependent and time-independent Schrödinger equations) exposed a different problem lurking at the heart of quantum mechanics by proposing a thought-experiment now known as the "Schrödinger's Cat" problem. In what follows, I will reframe his argument somewhat to fit with what we have studied.

The setup for the Schrödinger's Cat thought experiment

Suppose we place in an opaque and isolating box the following sequence of devices (see figure Q9b.3). When we push the button on the side of the box, the first device emits one electron in a quantum wavefunction consistent with being in a monochromatic beam (see section Q5.3). This electron falls on a mask with two slits, each with a proximity detector. Suppose that about 10% of the time, the electron will collapse to one of a large number of position eigenfunction that correspond to emerging from the top slit (10% of the time it goes through the bottom slit, and 80% of the time it is not registered at all, indicating that it has been absorbed by the mask). A signal from the top proximity detector electromagnetically releases a hammer that breaks a flask that releases cyanide gas into the box. Now suppose that we place a hapless cat in the box, close the lid, and press the button. After the electron has time to reach the mask, but before we have "measured" the state of the system, the electron remains in the superposition wavefunction which we might write as something like

$$\psi(x) = \psi_{top}(x) + \psi_{not}(x) \tag{Q9b.8}$$

where $\psi_{top}(x)$ is the sum of all position eigenfunctions that trigger the top detector and $\psi_{not}(x)$ is the sum of all the other states. But the chain of devices connects the cat's life to electron's position. Assuming that quantum mechanics describes the universe at *all* scales, we can schematically represent the quantum state of the entire system in the box (before that state collapses) as follows:

$$\psi = \psi_{top}\cdot\psi_{dead} + \psi_{not}\cdot\psi_{alive} \tag{Q9b.9}$$

since every electron position eigenfunction that appears in the sum ψ_{top} is connected to the quantum state of the cat being dead, and the remainder of the electron position eigenfunctions correspond to the cat being alive. Since about 10% of the position eigenfunctions are in the sum ψ_{top}, the cat must therefore in a quantum state where it is 10% dead.

Why don't we ever see a cat in a superposition of states?

The problem is that no one has *never* observed a cat to be a superposition of dead and alive quantum states. When we open the box, we will find the cat to be either alive *or* dead. So the state of the system must have collapsed

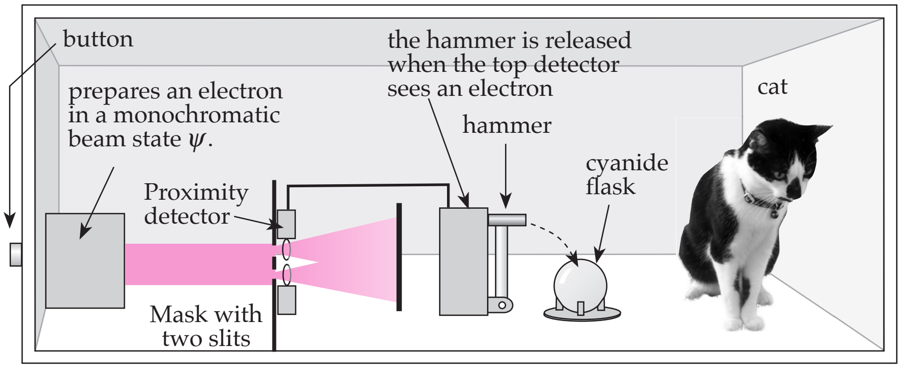


Figure Q9a.3
The "Schrödinger's cat" thought experiment. The electron is in a superposition of position eigenfunctions, with some (say, 10%) of those involved with going through the slit that triggers the hammer to fall. But because those outcomes are connected to the cat's death, total quantum state of the system involves a superposition of cat quantum states such that the cat is 10% dead.

some time between when we pushed the button and when we opened the box. But exactly *when* does the "measurement" (and thus the collapse) really take place? When the proximity detector fires? But until we open the box, do we actually *know* that the detector has fired? What device along the chain actually forces the collapse to happen?

Our classical intuition might lead us to say that the cat is *really* either alive or dead right after we push the button: we simply don't know which until we open the box. However our discussion of the question of whether the electron even has a meaningful position before it is measured should teach us to be suspicious of such an assertion. If the proximity detector were not there, the electron wavefunction must go through *both* slits in a two-slit interference experiment to contribute to the interference pattern. Now we introduce the proximity detector, but this only puts the combined system of electron plus proximity detector in the superposition state

$$\psi = \psi_{top} \cdot \psi_{fired} + \psi_{not} \cdot \psi_{quiet} \tag{Q9b.10}$$

where the ψ_{fired} and ψ_{quiet} here refer to the possible quantum states of the detector. What causes the quantum state of this (larger) quantum system to collapse? Exactly where along the chain is the collapse finally forced, and why does this happen?

The core issue that the Schrödinger experiment is designed to expose

The point is that we know that *microscopic* systems sometimes exist in superposition states, such as the broad wavefunction that creates the two-slit interference pattern. The problem is that we never observe *macroscopic* systems to have such states: we never see a cat that is both alive and dead. So we must have the phenomenon of wavefunction collapse to explain what we observe. But if we assume that the quantum model applies to all the universe, it is not clear at what level between the microscopic and the macroscopic the collapse occurs and why. This is the issue that Schrödinger's thought experiment was designed to expose.

General possible reasons we don't see superposition in macroscopic objects

The problem exposed remains unresolved. There are several general possible reasons that we don't see macroscopic objects in superposition states:

1. Quantum mechanics does not even in principle apply to macroscopic objects. (But why not, and at what size does classical physics take over?)
2. The system's quantum state spontaneously collapses at some stage of the process. (But at exactly what stage, and why?)
3. Something else is going on (but what and why)?

With regard to the first idea, let me note that experimental physicists have seen quantum superposition in increasingly large objects. Isolating macroscopic systems sufficiently from the environment so as to expose behaviors linked to superposition is very hard to do, but, quantum superposition *has* been observed in a microscopic vibrating object 1 μm thick and 40 μm long consisting of ~ 10^{13} atoms (O'Connell et. al. *Nature,* 464, April 1, 2010). This result does not absolutely contradict the first option, but does push the question of how large an object must be before it will behave classically.

The philosophical challenge that wavefunction collapse presents

The collapse phenomenon makes "measurement" in quantum mechanics radically different from measurement in classical physics. In classical physics, a measurement merely reveals (without modification) an observable's pre-existing value. But as we have seen, always-existing values for observables don't make sense in quantum mechanics, and the collapse of the wavefunction also implies that a "measurement" generally changes the quanton's state in a way that has empirical consequences. As Greenstein and Zajonc put it in their book *The Quantum Challenge* (Jones and Bartlett, 1977), measurement does not so much *reveal* reality as *create* it. The collapse phenomenon and the process of quantum "measurement" therefore carries a unique philosophical burden in quantum theory.

A sampling of possible ideas about the collapse of the wavefunction

There are many other proposed solutions to the collapse problem. To give you a sense of the range, consider the following. (1) Perhaps we can tweak the time-dependent Schrödinger equation to make quantum states actually collapse in suitably large systems (see Girardi, et al., *Phys. Rev. D.*, **34**, pp. 470-491, 1986). (2) Everett's "many worlds" proposal claims that the state does not collapse when a measurement is made; instead the universe splits into separate universes where each possibility is realized (see Everett, Rev. Mod. Phys, 29, pp. 454-462, 1957). (3) Perhaps human mind has some physical aspect that transcends quantum mechanics and which *causes* the collapse to occur, leaving true quantum mechanics with only one time evolution equation (Wigner, pp. 284-302 in *The Scientist Speculates*, Good, ed., Heinemann, 1962, and Penrose, *The Emperor's New Mind*, Oxford University Press, 1989). (4) Quantum Bayesians (QBists) reject the mainstream claim (by those whom QBists call "ψ-ontologists") that quantum wavefunctions represent physical reality. The collapse of the wavefunction is no problem if wavefunctions instead only represent the experimenter's personal knowledge: that knowledge can and does change irreversibly when new results become available (see Fuchs et. al. *Phys. Rev. A*, **65**, 022305 [2002] and von Baeyer, *Scientific American*, **308,** 6, June 2013). But how can an experimenter's personal knowledge affect whether an interference pattern exists or not?

As you can see, these proposals range from the credible to the truly bizarre. The absence of any scientific consensus on what to do with the collapse phenomenon makes it one of the great unsettled problems in the quantum model. This is all the more disturbing because of the crucial role that the rule plays in the theory's structure, as only through "measurement" does a theory connect with reality (whatever that is!). How can something so important be so problematic in a theory that is otherwise so wonderfully successful?

Is quantum mechanics the same as "necromancy"?

E. T. Jaynes wrote in *Foundations of Radiation Theory and Quantum Electrodynamics* (A. O. Barut, ed., Plenum, 1980) the following: "… present quantum theory not only does not use – it does not even dare to mention – the notion of a 'real physical situation.' Defenders of the theory say that … recognition of this constitutes deep new wisdom about the nature of human knowledge. I say that it constitutes a violent irrationality, that somewhere in this theory the distinction between reality and our knowledge of reality has become lost, and the result has more the character of medieval necromancy than science."

Perhaps one of you will help resolve these unsolved problems, and thus help answer Jaynes's trenchant critique of quantum mechanics.