C8 Conservation of Energy

The two-page chapter opener provides an overview before starting the chapter and summary for review.

Chapter Overview

Section C8.1: Introduction to Energy

A system's **total energy** E is the sum of the **kinetic energies** of its constituent fundamental particles and the **potential energies** of the interactions between them:

$$E = K_1 + K_2 + K_3 + \ldots + V_{12} + V_{13} + \ldots + V_{23} + \ldots \tag{C8.2}$$

Like momentum, the total energy of an isolated system is conserved. However, energy is unlike momentum and angular momentum in that (1) energy is an *ordinary number*, not a vector, and (2) interactions as well as objects can contain energy.

Section C8.2: Kinetic Energy

Formula boxes highlight important equations, define their symbols, and link them with their purposes and limitations.

$$K = \tfrac{1}{2}m|\vec{v}|^2 \quad \text{or equivalently,} \quad K = \frac{1}{2}\frac{(m|\vec{v}|)^2}{m} = \frac{|\vec{p}|^2}{2m} \tag{C8.3}$$

- **Purpose:** This equation describes the kinetic energy K that a particle of mass m contains when it is moving with speed $|\vec{v}|$ or with momentum $\vec{p}$.
- **Limitations:** This equation is an approximation that is accurate when the particle's speed $|\vec{v}| << c$, where c is the speed of light. It also applies to non-rotating objects modeled as particles if $|\vec{v}|$ is the speed of the object's center of mass.

Energy is measured in **joules**, where $1\text{ J} = 1\text{ kg·m}^2/\text{s}^2$, which are clearly the units of K.

When a low-mass object interacts with a much more massive object (such as the earth) initially at rest, we can ignore the massive object's kinetic energy.

Section C8.3: Potential Energy

A fundamental interaction's potential energy depends only on the interaction, the characteristics of the interacting particles, and the separation r between the particles.

However, for reasons discussed in the section, we can only define an interaction's potential energy up to an overall position-independent constant. We usually specify this constant by specifying an **reference separation** where $V \equiv 0$. An interaction's potential energy can thus be *negative* in a situation where it has less energy than it would have at the reference separation. Even a system's total energy E can be negative if the system has less energy in a situation than it would with all its particles at rest at their reference separations.

The section illustrates how by choosing a different reference separation, we can adapt the large-scale gravitational potential energy formula to make it more useful for objects moving near the earth's surface.

Section C8.4: Fundamental Potential Energy Formulas

Of the four fundamental interactions in the Standard Model of particle physics, only the gravitational and electromagnetic interactions act over macroscopic distances. Potential energy formulas for the gravitational interaction and the electrostatic aspect of the electromagnetic interaction are as follows:

$$V_g(r) = -G\frac{m_1 m_2}{r} \qquad \begin{bmatrix}\text{reference separation:}\\ V_g(r) \equiv 0 \text{ when } r = \infty\end{bmatrix} \tag{C8.12}$$

- **Purpose:** This is the potential energy function $V_g(r)$ for the gravitational interaction between two particles or spherical objects whose masses are m_1 and m_2 and which are separated by a (center-to-center) distance r. G is the **Universal Gravitational Constant**: $G = 6.67 \times 10^{-11}\text{ N·m}^2/\text{kg}^2$.
- **Limitations:** This equation technically applies only to particles and/or spherical objects *at rest*. But it is an excellent approximation if the interacting objects' speeds are small compared to that of light.

Wide margins give students a place to write notes and questions.

$$V_g(z) = m|\vec{g}|z \qquad \begin{bmatrix}\text{reference separation:}\\ V_g(z) = 0 \text{ when } z = 0\end{bmatrix} \tag{C8.13}$$

- **Purpose:** This is the potential energy function $V_g(z)$ for the gravitational interaction between the earth and a small object of mass m near the earth's surface, where z is the small object's vertical position and $|\vec{g}| = 9.8\text{ m/s}^2$ is the gravitational field strength at the earth's surface.
- **Limitations:** This is an *approximation* that is sound when all positions z of interest (including the point where $z = 0$) are within ~50 km of the earth's surface, and the object's speed is small compared to that of light.

$$V_e(r) = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{q_1 q_2}{r} \qquad \begin{matrix}\text{reference separation:}\\ V_e(r) \equiv 0 \text{ when } r = \infty\end{matrix} \tag{C8.14}$$

- **Purpose:** This is the potential energy function $V_e(r)$ for the electrostatic part of the electromagnetic interaction between two particles or uniformly charged spherical objects whose charges are q_1 and q_2 and which are separated by a (center to center) distance r. We call the constant $1/4\pi\varepsilon_0$ (which we write that way for historical reasons) the **Coulomb constant**: $1/4\pi\varepsilon_0 = 8.99 \times 10^9\text{ N·m}^2/\text{C}^2$.
- **Limitations:** This equation strictly applies only to particles (or spherical objects) *at rest*, but is a good approximation as long as the particles' speeds are small compared to the speed of light.

Section C8.5: Internal Energy and Power

As we will see in later chapters, a macroscopic object can contain **internal energy** U in "hidden" forms. In this chapter we will assume that these energies do not change and thus cancel out of the conservation of energy master equation.

Power is the *rate* at which energy is transported or converted in a physical process. It is measured in **watts**, where $1\text{ W} = 1\text{ J/s}$.

Section C8.6: Isolation

A system's energy is conserved only if it is isolated. In general, we must ensure that a system is not only isolated in one of the ways required for momentum conservation but also that energy cannot flow into or out of the system in some hidden form, but in this chapter we will consider situations where this does not happen.

Section C8.7: Solving Conservation-of-Energy Problems

This section presents several example problem solutions. These examples illustrate that when constructing models for conservation of energy problems, we have to consider how we are going to handle each *internal* interaction (something we did not have to worry about with conservation of momentum or angular momentum).

C8.1 Introduction to Energy

Definitions of potential and kinetic energy

Sidebar notes help readers track the argument's flow and find things later.

Energy is one of the richest and most productive ideas in all of physics. It is also more subtle than the other conservation laws: while Newton himself knew that interactions conserve momentum, it was not until the 1840s that physicists realized fully that energy was also universally conserved. My goal in this section is to introduce the concept of energy as it applies to elementary particles and fundamental interactions, where the model is simplest.

Boldface letters highlight technical terms when they first appear, close to their definitions.

In addition to momentum and angular momentum, interactions transfer *energy*, expressing the symmetry that the laws of physics do not depend on what time it is. But though only *particles* can contain significant amounts of momentum or angular momentum,* *interactions* can contain significant amounts of energy, which we call the interaction's **potential energy** V. This potential energy is an *ordinary number* (not a vector) that depends on the type of interaction and the particles' physical separation. Each particle also contains a certain amount of **kinetic energy** K, a ordinary number that increases as a particle's speed increases. Any change in the interacting particles's separation leads to a *transformation* of energy from the interaction's potential energy V to the particles' kinetic energies K_1 and K_2 or vice versa. Because energy is transformed (not created or lost), a two-particle system's **total energy**

$$E = K_1 + K_2 + V \tag{C8.1}$$

is conserved. All of the Standard Model's core interactions behave this way.

Figure C8.1 illustrates this idea in the context of two particles floating in space that attract each other gravitationally (which is one of the Standard Model's core interactions). The potential energy of a gravitational interaction happens to decrease as the particles get closer, so interaction causes the particles' speeds to increase at just the right rate so that the increase in their kinetic energies balances the decrease in the interaction's potential energy.

Kinetic, potential, and total energy in a multi-particle system

If a system contains more than two interacting particles, its total energy (as one might guess) is the sum of *all* the particle's kinetic energies and the potential energies of *all* the system's internal interactions:

$$E = K_1 + K_2 + K_3 + \ldots + V_{12} + V_{13} + \ldots + V_{23} + \ldots \tag{C8.2}$$

where V_{12} is the potential energy of the interaction between particles 1 and 2, V_{13} is the same between particles 1 and 3 and so on. (If more than one interaction acts between a given pair of particles, we should include a potential energy term for each interaction. This simple definition of the total energy applies whenever the newtonian approximation applies (otherwise the definition becomes a bit more complex).

Energy, like momentum, is conserved

Because a system's *internal* interactions only involve transformations between kinetic and potential energies that appear in this sum, the system's total energy is not changed by such interactions. But *external* interactions can affect the kinetic energies of the system's particles and so *transfer* energy into or out of the system. A system's total energy E (like its total momentum and total angular momentum) is therefore conserved only if the system is *isolated*.

But energy is different than momentum in several ways

However, energy is different than momentum and angular momentum in several important ways. First, note that E, K, and V are just *numbers*, while momentum $\vec{p}$ (and angular momentum $\vec{L}$) are *vectors*. Second, because interactions do *not* store momentum or angular momentum but *can* store potential energy, internal interactions do *not* contribute to a system's total momentum or angular momentum but *do* contribute to its total energy. Finally, though we can always determine a system's total ordinary momentum from the velocity of its center of mass or its angular momentum from its rotation rate, we will see in subsequent chapters that a system can store energy in "hidden" forms not linked to the macroscopic motion of the system's parts.

*Actually, interactions *do* store momentum and angular momentum but for reasons beyond our scope here, the amounts stored are significant only in unusual situations, and in those cases, other models are typically more useful.

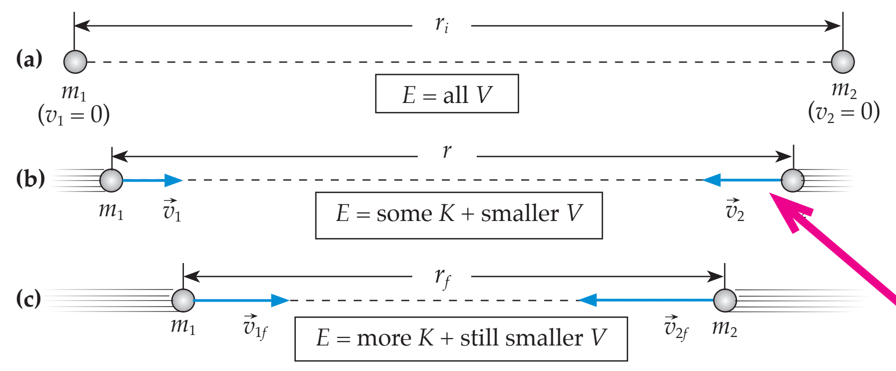


Figure C8.1
The gravitational interaction between two particles transforms some of its potential energy to particle kinetic energy without changing the system's total energy.

Figures use color to highlight certain elements (often, as here, to indicate items that exist only in one's mind).

C8.2 Kinetic Energy

The kinetic energy that a moving object contains

Physicists in the 18th and 19th centuries found that the previous section's model works experimentally if we assume that a particle's kinetic energy is

$$K = \tfrac{1}{2} m|\vec{v}\,|^2 \quad \text{or equivalently,} \quad K = \tfrac{1}{2}\frac{(m|\vec{v}\,|)^2}{m} = \frac{|\vec{p}\,|^2}{2m} \tag{C8.3}$$

- **Purpose:** This equation describes the kinetic energy K that a particle of mass m contains when it moves at speed $|\vec{v}\,|$ or with momentum $\vec{p}$.
- **Limitations:** This equation is an approximation that is useful when $|\vec{v}\,| << c$, where c is the speed of light. It also applies to a non-rotating macroscopic object if $|\vec{v}\,|$ is the speed of the object's center of mass.

Wide margins continue in the body of the chapter (but not in the homework problems section)

The text uses this unambiguous notation for vector magnitudes throughout.

Equation C8.3 implies that the SI units of kinetic energy (and thus all other forms of energy) must be kg·m^2/s^2. Since energy is such an important concept in physics, it has it own SI unit, the **joule**, where

$$1\text{ J} \equiv 1\text{ joule} \equiv 1\text{ kg}\cdot\text{m}^2/\text{s}^2 \tag{C8.4}$$

All equations (no matter how minor) have numbers.

roughly the kinetic energy of a half-gallon carton of milk moving at 1 m/s.

Consider now an isolated system consisting of two particles (or macroscopic objects that we can model as particles). Suppose that the particles have masses m_1 and m_2 and are initially at rest in some frame. The particles then interact. Since the system's total momentum is initially zero and must be conserved, if the first particle ultimately gets momentum $\vec{p}_1$ from the interaction the other must get $\vec{p}_2 = -\vec{p}_1$. Equation C8.3 then implies that

$$\frac{K_{2f}}{K_{1f}} = \frac{|\vec{p}_2|^2}{\cancel{2}m_2}\frac{\cancel{2}m_1}{|\vec{p}_1|^2} = \frac{\cancel{|-\vec{p}_1|^2}}{\cancel{|\vec{p}_1|^2}}\frac{m_1}{m_2} = \frac{m_1}{m_2} \tag{C8.5}$$

So, while the internal interaction always gives the particles *equal* magnitudes of momentum (assuming they start from rest), it gives each a kinetic energy *inversely proportional to its mass*. So if $m_1 << m_2$, then $K_2 << K_1$. This means that

C8.5 Internal Energy and Power

Internal energy

The model described so far in this chapter is adequate for describing interactions between *particles*. But interacting macroscopic objects can also store energy internally in "hidden" forms that are independent of those objects' center-of-mass kinetic energies or the potential energies due to the objects' separations. For example, all macroscopic objects contain *thermal energy* that may become involved in an interaction.

We will explore such internal energies in future chapters. For now it is enough to know that a macroscopic object can contain an **internal energy** U in addition to its kinetic energy K. In this chapter, we will focus on situations where objects' internal energies do not change significantly and so can be ignored. Gravitational or electrostatic interactions between macroscopic objects typically do *not* couple to those objects' internal energies unless the objects are significantly and/or suddenly deformed by the interaction.

The concept of *power*

Now, recall that *force* is the rate at which an interaction delivers impulse (that is, transfers momentum) and *torque* is the rate at which an interaction delivers twirl (that is, transfers angular momentum). The analogous concept for energy is *power*: the rate at which an interaction changes a particle's kinetic energy is the **power** P that the interaction delivers to that particle.

However, energy is different from momentum and angular momentum not only because energy is an ordinary number instead of being a vector (implying that power is also an ordinary number) but also because energy can exist a variety of forms (kinetic, potential, and various kinds of internal energy). Therefore, physicists typically broaden the definition of "power" to describe the rate of *any* kind of energy transfer across a system boundary, even if it is *not* mediated by a macroscopic force-exerting interaction. We will talk about this more in future chapters.

The standard SI unit of power is the **watt**, where 1 W $\equiv$ 1 J/s. An older unit is the **horsepower**, which we now define as follows: 1 hp $\equiv$ 746 W. This means that a 100-hp automobile engine is able to transform a "hidden" form of energy in the car's fuel (chemical energy) and transfer it across the engine's boundary to the rest of the car (in a form that can be converted to the car's kinetic energy) at a rate of 74,600 joules per second.

C8.6 Isolation

A system's total energy is conserved only if the system is *isolated*, that is, if no net energy flows across its boundary. In the case of momentum conservation, we called a system "isolated" if (1) it floats in space, (2) is functionally isolated (it experiences no *net* momentum flow from external interactions), or (3) involves a short-duration "collision." These categories are still helpful when considering conservation of energy, but with some complications.

Hidden energy flows can make determining isolation tricky

These complications arise because in systems involving macroscopic objects, energy can sometimes flow across a boundary in "hidden" forms (for example, heat). Such flows may not involve significant net momentum transfers, so a system can be functionally isolated with regard to momentum flows but not with regard to energy flows. Even a system that floats in space (such as a star) may not be isolated if it radiates a lot of heat.

Again, we will in this chapter consider only situations where such issues do not arise. Systems of objects involving gravitational and/or long-range electromagnetic interactions that are isolated with regard to momentum flows are also almost always isolated with regard to energy flows.

C8.7 Solving Conservation-of-Energy Problems

Differences between momentum and energy checklists

This section will illustrate how we can use the information in this chapter to solve basic conservation-of-energy problems. The task checklist for such problems is mostly similar to the checklist for other conservation problems, but with two modifications. First, since energy is an *ordinary number,* we rarely need to resolve vectors into components, so drawing coordinate axes is often not necessary. Secondly, because internal interactions can in principle store energy and/or channel energy into internal energies, they take on a significance in conservation-of-energy problems that they do not have in conservation-of-momentum or conservation-of-angular-momentum problems.

List each interaction and *how you will handle it*

Therefore a valuable part of one's model in a conservation-of-energy problem solution is a list of all the system's significant internal interactions and *how you intend to handle each interaction* in your model. In the context of this chapter "handling" an interaction means specifying what potential energy function you will use to describe the interaction's energy. For example, in the case of a gravitational interaction, you might say "Gravitational: $V_g(z) = m|\vec{g}|z$." In the case of an electrostatic interaction you might say: "Electrostatic: $V_g(r) = q_1q_2/(4\pi\varepsilon_0 r)$." Creating such a list nudges you to think carefully about each internal interaction and its energy implications and efficiently communicates your thinking about those interactions to your reader. Such lists will become even more important later.

If you ever use reference positions for potential energies that are *not* the conventional positions listed in equations C8.12 through C8.14, be *sure* to describe these carefully, so that your reader does not become confused.

So the conservation-of-energy problem checklist looks like this:

Problem-solving checklists in units C and N guide students in solving various types of problems.

Conservation-of-energy problem checklist

CoE-Specific Checklist
- ❐ Describe the system.
- ❐ Describe its external interactions.
- ❐ Describe how it is isolated.
- ❐ List its internal interactions and state how to handle each.
- ❐ Draw "initial/final" diagrams.
- ❐ Describe non-standard ref. positions.

General Checklist
- ❐ Define symbols.
- ❐ Draw a picture.
- ❐ Describe model & assumptions
- ❐ Use a master equation.
- ❐ List knowns and unknowns.
- ❐ Do algebra symbolically.
- ❐ Track units. ❐ Check result.

Master equation for conservation-of-energy problems

The master equation for conservation of energy problems is simply $E_f = E_i$, or in a bit more detail,

$$K_{1i} + K_{2i} + \ldots + U_{1i} + U_{2i} + \ldots + V_{Ai} + V_{Bi} + \ldots$$
$$= K_{1f} + K_{2f} + \ldots + U_{1f} + U_{2f} + \ldots + V_{Af} + V_{Bf} + \ldots \quad \text{(C8.15)}$$

where K_{ni} and K_{nf} are the nth object's initial and final kinetic energies respectively, U_{ni} and U_{nf} are its initial and final internal energies, respectively, and V_{Ai} and V_{Af} are interaction A's initial and final potential energies, V_{Bi} and V_{Bf} are the same for interaction B, and so on. "Using a master equation" means adapting the abstract form of the equation given above to the given situation.

In this chapter, we will assume that the interacting objects' internal energies do not change. This means that all the U's on one side of the equation will cancel those on the other.

The examples on the following page illustrate solutions that are consistent with the checklists described above.

Colored comments in units C and N indicate features of a good solution and how the solution satisfies the checklists.

Example C8.1

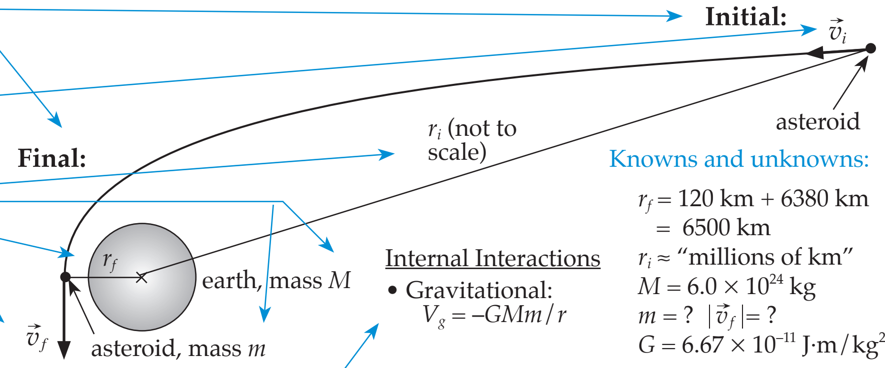
Problem: Astronomers spot an asteroid that is currently several million kilometers from the earth (but about the same distance from the sun as the earth is) moving at 12 km/s relative to the earth. When they predict the asteroid's future trajectory, they realize that the asteroid will pass within 120 km of the earth's surface, just grazing the outer atmosphere (whew!). What will be the asteroid's speed as it passes?

Solution Initial and final diagrams for this situation appear below:

Initial and final diagrams

Symbol definitions

List of internal interactions with potential energy formulas



Internal Interactions
- Gravitational: $V_g = -GMm/r$

Knowns and unknowns:

$r_f = 120\text{ km} + 6380\text{ km}$
$= 6500\text{ km}$
$r_i \approx$ "millions of km"
$M = 6.0 \times 10^{24}\text{ kg}$
$m = ?\quad |\vec{v}_f| = ?$
$G = 6.67 \times 10^{-11}\text{ J}\cdot\text{m/kg}^2$

System description

How system is isolated, with justification

Discussion of how the internal interaction will be handled

More symbol definitions

Approximations and assumptions stated

The system here is the earth and the asteroid, which is isolated because it floats in space. (Note that though the system is initially several million kilometers across, it is much farther than that from the sun, the most significant source of external gravitation.) The only internal interaction is a gravitational interaction between the earth and asteroid, which we will handle with the universal gravitational interaction potential energy formula. This interaction will not affect the internal energies of either the earth or asteroid, so their internal energies U_e and U_a (respectively) remain fixed. Since the earth is much more massive than even an asteroid, its kinetic energy K_e in this process will remain essentially zero. The master equation in this case then becomes

Master equation

$$\tfrac{1}{2}m|\vec{v}_i|^2 + \cancel{U_a} + \overbrace{K_{ei}}^{=0} + \cancel{U_e} - \overbrace{\frac{GMm}{r_i}}^{\approx 0} = \tfrac{1}{2}m|\vec{v}_f|^2 + \cancel{U_a} + \overbrace{K_e}^{\approx 0} + \cancel{U_e} - \frac{GMm}{r_f} \qquad \text{(C8.16)}$$

The problem only vaguely states the value of r_i, but "several million kilometers" is at on the order of $1000r_f$, so $GMm/r_i \approx (GMm/r_f)/1000$, so we can treat $GMm/r_i \approx 0$ in comparison to GMm/r_f, as indicated above. So, throwing away the terms that are zero, nearly zero, or cancel out, we have

Algebra with symbols

$$\tfrac{1}{2}\cancel{m}|\vec{v}_i|^2 = \tfrac{1}{2}\cancel{m}|\vec{v}_f|^2 - \frac{GM\cancel{m}}{r_f} \;\Rightarrow\; |\vec{v}_f|^2 = |\vec{v}_i|^2 + \frac{2GM}{r_f} \;\Rightarrow\; |\vec{v}_f| = \sqrt{|\vec{v}_i|^2 + \frac{2GM}{r_f}}$$

Calculation includes and tracks units

$$\Rightarrow\; |\vec{v}_f| = \sqrt{\left(12{,}000\,\frac{\text{m}}{\text{s}}\right)^2 + 2\left(6.67\times10^{-11}\,\frac{\cancel{\text{N}}\cdot\text{m}^2}{\cancel{\text{kg}}^2}\right)\frac{6.0\times10^{24}\,\cancel{\text{kg}}}{6{,}500{,}000\,\cancel{\text{m}}}\left(\frac{1\,\cancel{\text{kg}}\cdot\cancel{\text{m}}/\text{s}^2}{1\,\cancel{\text{N}}}\right)}$$

$$= 1.63\times10^4\,\frac{\text{m}}{\text{s}} = 16.3\,\frac{\text{km}}{\text{s}} \qquad \text{(C8.17)}$$

Final check of plausibility

Note that the units of both terms in the square root end up being m^2/s^2 (as they must). The answer is also plausible (larger than the initial 12 km/s). Note also that the asteroid's unknown mass m cancels out without requiring another equation to determine it. (For future reference, this is common in problems involving gravitation.)

The layout pays close attention to what appears on facing pages, so that a reader does not have to turn a page forward or backward to see (for example) a diagram needed to understand the sentences he or she is reading.

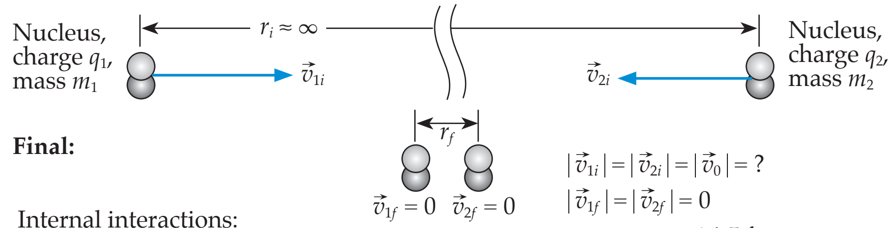
Fewer but more important exercises allow students to practice important skills while reading (answers appear at the end of the chapter).

Example C8.2

The frame around each example now more clearly indicates what is inside and what is outside the example's boundaries.

Problem: A deuterium nucleus consists of a proton bound to a neutron. If we can get two such nuclei close enough together so that they almost touch (about 1.5 fm = 1.5×10^{-15} m center-to-center), they can fuse to form a helium nucleus. However, since they have the same positive charge, electrostatic repulsion seeks to keep them apart. Suppose that we fire two such nuclei directly toward each other with the same initial speed $|\vec{v}_0|$. What minimum value must $|\vec{v}_0|$ have (as a fraction of the speed of light) for the nuclei to fuse?

Solution Initial and final diagrams for this situation appear below:



Internal interactions:
- Electrostatic:
$V_e(r) = \left(\dfrac{1}{4\pi\varepsilon_0}\right)\dfrac{q_1q_2}{r}$

$|\vec{v}_{1i}| = |\vec{v}_{2i}| = |\vec{v}_0| = ?$
$|\vec{v}_{1f}| = |\vec{v}_{2f}| = 0$
$r_i \approx \infty,\quad r_f \le 1.5\text{ fm}$
$q_1 = q_2 = e \equiv +1.60\times10^{-19}\text{ C}$
$m_1 = m_2 = (1.67 + 1.68)\times10^{-27}\text{ kg}$
$= 3.75\times10^{-27}\text{ kg}$

The system here is the pair of nuclei. Let's assume that this whole process happens so quickly that we can consider it to be a "collision." The nuclei interact gravitationally as well as electrostatically, but the gravitational interaction is negligible in comparison (see problem C8D.3). Note that even if we fire the nuclei from only a few millimeters apart, r_i is still enormously larger than 1.5 fm, so we can consider $r_i \approx \infty$. The nuclei will reach their smallest possible separation r_f when the system's energy is entirely potential, that is, when $\vec{v}_{1f} = \vec{v}_{2f} = 0$. We will assume that the nuclei's internal energies U_1 and U_2 don't change. The master equation for this situation therefore reads

$$\tfrac{1}{2}m_1|\vec{v}_{1i}|^2 + \cancel{U_1} + \tfrac{1}{2}m_2|\vec{v}_{2i}|^2 + \cancel{U_2} + \overbrace{\frac{q_1q_2}{4\pi\varepsilon_0 r_i}}^{\approx 0} = \overbrace{K_{1f}}^{0} + \cancel{U_1} + \overbrace{K_{2f}}^{0} + \cancel{U_2} + \frac{q_1q_2}{4\pi\varepsilon_0 r_f}$$

$$\Rightarrow\; 2\left(\tfrac{1}{2}m|\vec{v}_0|^2\right) = \frac{e^2}{4\pi\varepsilon_0 r_f} \;\Rightarrow\; \left|\frac{\vec{v}_0}{c}\right|^2 = \frac{e^2}{4\pi\varepsilon_0 mc^2 r_f} \qquad \text{(C8.18)}$$

$$\Rightarrow\; \left|\frac{\vec{v}_0}{c}\right| = \frac{e}{c}\sqrt{\frac{1}{4\pi\varepsilon_0 m r_f}} \ge \frac{1.60\times10^{-9}\,\cancel{\text{C}}}{3.00\times10^8\,\text{m/s}}\sqrt{\frac{8.99\times10^9\,\text{J}\cdot\cancel{\text{m}}/\cancel{\text{C}}^2}{(3.75\times10^{-27}\,\text{kg})(1.5\times10^{-15}\,\cancel{\text{m}})}}$$

$$= \frac{0.023}{\cancel{\text{m/s}}}\sqrt{\frac{\cancel{\text{J}}}{\cancel{\text{kg}}}\left(\frac{1\,\cancel{\text{kg}}\cdot\cancel{\text{m}^2/\text{s}^2}}{1\,\cancel{\text{J}}}\right)} = 0.023 \qquad \text{(C8.19)}$$

The result is unitless (as expected) and only 2.3% of the speed of light, so the potential energy formula's assumption that $|\vec{v}_{1,2}| << c$ is reasonably satisfied.

Exercise C8X.3

Add comments to the above solution (similar to those in the previous example) explaining how it satisfies the conservation-of-energy problem checklist. Also see if you can find and fix a few subtle omissions.