POWER-LAW FITTING AND LOG-LOG GRAPHS

"She had taken up the idea, she supposed, and made everything bend to it."
--- *Emma*

5.1 DEALING WITH POWER LAWS

Although many relationships in nature are linear, some of the most interesting relationships are nonlinear. Power-law dependences, of the form

$$y(x) \ = \ kx^n \tag{5.1}$$

are particularly common. In many cases, we might suspect that two experimental variables are related by a power-law relationship, but are unsure of what k or n are. For example, we might suspect that the period T of a simple harmonic oscillator might depend on the mass m of the oscillating object in some kind of power-law relationship, but we might be unsure of exactly what the values of either n or k. If we knew n, then we could plot y vs. x^n to get a straight line; the slope of that line would then be k. But if we *don't* know n, it would seem at first glance that the best we could do would be simply try different possible values of n to see what works. This could get old fast.

We can, however, take advantage of the properties of logarithms to convert *any* relationship of the type given by equation 5.1 into a linear relationship, even if we do not know either k or n! The most basic property of logarithms (for any base, but let's assume base-ten logs) is that:

$$\log(ab) \ = \ \log a + \log b \tag{5.2a}$$

From this it follows that

$$\log(a^n) \ = \ \log(\overbrace{a \cdot a \cdot \ldots \cdot a}^{n \text{ times}}) \ = \ \overbrace{\log a + \log a + \ldots + \log a}^{n \text{ times}} \ = \ n \log a \tag{5.2b}$$

(this is actually true even for non-integer n). Also, in the case of base-ten logarithms, this means

$$\log(10^b) \ = b \log(10) \ = \ b \cdot 1 \ = \ b \tag{5.2c}$$

which shows that raising 10 to a power is the *inverse* operation to taking the (base-ten) log.

With this in mind, let us take the (base-ten) logarithm of both sides of equation 5.1 and use the properties described by equation 5.2. If we do this we get

$$\log y \ = \ \log(kx^n) \ = \ \log k + \log x^n \ = \ \log k + n \log x \tag{5.3}$$

Now define $\log x \equiv u$ and $\log y \equiv v$. Substituting these into 5.3 and rearranging, we get

$$v = nu + \log k \tag{5.4}$$

Now, *this* is the equation of a straight line. This means that if we graph v vs. u (that is, log y vs. log x), we should end up with a straight line, even if we do not know what n and k are.

Furthermore,the slope of this line is the unknown exponent n in equation 5.1. We can therefore *find* the value of n by calculating the slope in the usual way. That is,

$$n \ = \ \frac{\Delta v}{\Delta u} \ = \ \frac{v_2 - v_1}{u_2 - u_1} \ = \ \frac{\log y_2 - \log y_1}{\log x_2 - \log x_1} \tag{5.5}$$

The value of the intercept (which is the value of $v = \log y$ when $u = \log x = 0$) is $\log k$, so if we can find the intercept and its uncertainty, we can find k and its uncertainty.

In summary, we can take *any* relationship of the form given in equation 5.1, take the logarithm of both sides, and convert it to a linear relationship whose slope and intercept are related to the unknown values of n and k. This is, therefore, a very powerful way of learning about unknown power-law relationships (and displaying them). A graph that plots $\log y$ versus $\log x$ in order to linearize a power-law relationship is called a **log-log graph**.

5.2 AN EXAMPLE OF A LOG-LOG GRAPH

As an example, consider a hypothetical experiment testing how the period of an object oscillating at the end of a spring depends on the object's mass. Table 5.1 gives a set of measurements taken during this experiment, Figure 5.1 shows a graph of period vs. the mass. Table 5.2 shows the base-10 logs of the measurement data, while Figure 5.2 shows a graph of $\log T$ vs. $\log M$.

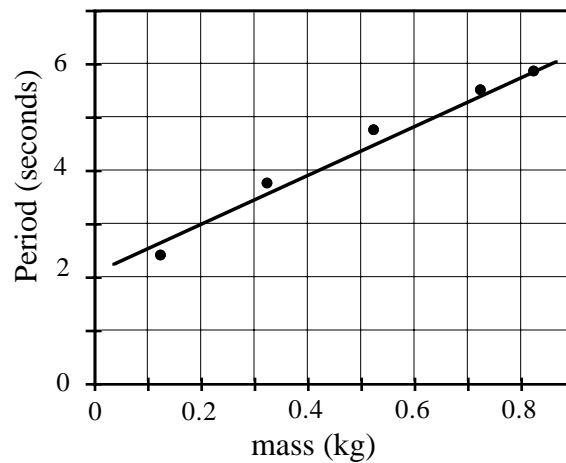| Table 5.1 || Table 5.2 ||
mass (M)	period (T)	$\log M$	$\log T$
0.125 kg	2.42 s	−0.903	0.384
0.325 kg	3.76 s	−0.488	0.575
0.525 kg	4.75 s	−0.280	0.677
0.725 kg	5.52 s	−0.140	0.742
0.825 kg	5.87 s	−0.084	0.769

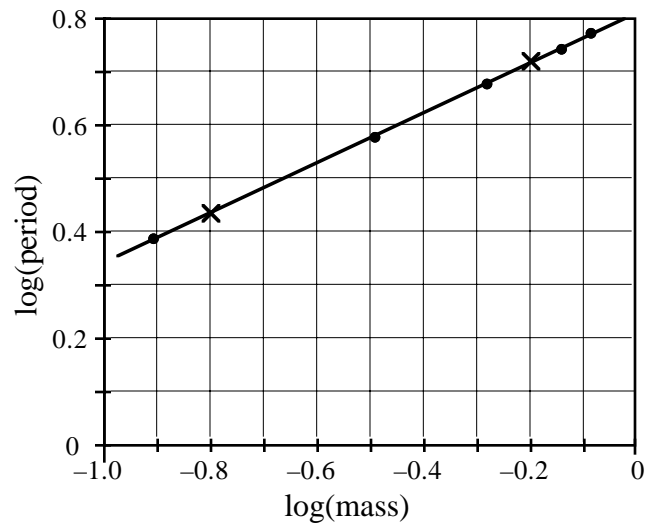**Figure 5.1:** Graph of the oscillation period as a function of mass.

Figure 5.2: Graph of the log of the oscillation period as a function of the log of the mass.

We can see that the graph of the period vs. the mass does *not* yield a very good straight line: (if the uncertainties are smaller than the dots representing the data points, we would have to say that a straight line is inconsistent with the data). On the other hand the plot of $\log T$ vs. $\log M$ is a very nice straight line, suggesting that the period T and the mass M have a power-law relationship.

What are n and k according to this experiment? We can easily get a quick estimate from the graph. For the sake of round numbers, consider the points marked with ✕s on the graph above. The slope of this graph (rise over run) is thus

$$n = \frac{\log T_2 - \log T_1}{\log M_2 - \log M_1} = \frac{0.72 - 0.43}{0.80 - 0.20} = 0.48 \qquad (5.6)$$

(The two points that you choose to compute the slope need not correspond to *actual* data points: simply choose convenient points on your drawn line near the ends of the line. I chose the points so that the denominator of the expression above would be simple.) Since exponents in physical situations are most often integers or simple fractions, we might guess that the actual value of n is 1/2 (this turns out to be the theoretical value as well as we will see chapter N11 in the class text).

The intercept is the place where the line crosses the $\log M = 0$ grid line. According to the graph, this is roughly where $\log T = 0.82$ (note that the $\log M = 0$ line is the *right* edge of the graph here, not the left!). So the intercept is $0.82 = \log k$, which means that $k = 10^{0.82} = 6.6$ s/kg$^{1/2}$.

5.3 A COMMENT ABOUT LOGARITHMS AND UNITS

You may also have noticed that units seem to spring in and out of these calculations rather haphazardly. This behavior conflicts with a general rule about special functions: the log function, like sine, cosine, and the exponential function, is supposed to have a dimensionless arguments. That is, the number you take the logarithm of is supposed to be a pure number, with *no* units or dimensions (like seconds or centimeters), attached to it. What's going on?

One can think of a number with units as the product of the number part and the units part; that is, 10 cm is 10 (a pure number) multiplied by the unit of centimeters. When you take the logarithm of a number with units, then, you are taking the logarithm of this product, which behaves like the logarithm of all products: the result is the sum of the logarithms of the two numbers making up the product. The logarithm of 10 cm is therefore $\log (10) + \log$ cm $= 1 + \log$ cm. You can't give a numerical result for $\log$ cm, which may seem rather disturbing.

Fortunately, this all works out fine anyway, because you typically find the logarithm of a quantity with units only when you in the process of finding the difference between two logarithms of quantities with the same units. Consider, for example, the situation where we compute the slope using equation 5.6. If we attach to each logarithm in the numerator the appropriate term of $\log$ s (where s is the unit of seconds here), we see that two $\log$ s terms cancel out when we subtract, leaving a unitless number in the numerator. The same thing happens in the denominator. Therefore, the slope ends up being a unitless number, as it should be.

To find the units of k, note that equation 5.4 in this situation really *ought* to read

$$\log \tau + \log \text{ s} = n \log \mu + n \log \text{ kg} + \log k \qquad (5.7)$$

(where in this expression we are considering τ and μ to be the unitless parts of T and M respectively). The intercept is where the line crosses the vertical grid line corresponding to $\log \mu = 0$: we see from the graph that $\log \tau$ has the value $\log \tau_0 = 0.82$ there. Therefore equation 5.7 implies that

$$\log \tau_0 + \log \text{ s} = n \log \text{ kg} + \log k \quad \Rightarrow \quad \log k = \log \tau_0 + \log \text{ s} - n \log \text{ kg} \qquad (5.8)$$

When we take the antilog of (that is, 10 to the power of) both sides of this, all the items in logs get multiplied together, so we get (assuming that n is really 1/2):

$$k = \frac{10^{\log \tau_0} \cdot \text{s}}{\text{kg}^n} = \frac{10^{0.82} \cdot \text{s}}{\text{kg}^{1/2}} = 6.6 \frac{\text{s}}{\text{kg}^{1/2}} \qquad (5.9)$$

Keeping track of these unit terms when working with logarithms involves a *lot* of work, however, and less often pays off the way that keeping track of units in normal equations does.

Therefore, people generally *ignore* the units associated with logarithmic quantities, and fill in the units of quantities after taking the antilog (as I did with k in the last section) to make them consistent across the master equation 5.1. But if you ever get *confused* about units and want to make *sure* that things work out correctly, this is how to do it.

5.4 A PROCEDURE FOR EXPLORING POWER-LAW RELATIONSHIPS

Log-log graphs are most useful when you *suspect* your data has a power-law dependence and you want to test your suspicion. Sometimes your suspicion is based on a theoretical prediction, sometimes a previous low-level Cartesian plot. Figures 5.3 through 5.5 are typical Cartesian graphs that could be power laws. Whatever the source of your suspicion, your next step is to plot the logarithms of your data as a low-level graph. If this graph looks like a pretty good straight line (within your experimental uncertainties) you can proceed to the next steps.

Once you have plotted the points, you should use a ruler to draw the straight line that you think best fits your data. You can then use this line to estimate the slope n using equation 5.6. You can also estimate the value of the constant k in equation 5.1 by extrapolating your straight line back to the vertical grid line where the value of the independent variable (let's call it x) is equal to 1 (and thus $\log x = 0$): the value of log k is the vertical scale reading where your best fit line crosses this vertical grid line where $\log x = 0$. Note that this vertical line may not correspond to the left edge of your graph! In Figure 5.2, for example, it happens to be at the *right* edge of the graph, and on a general log-log graph, it could be almost anywhere. (If the line $\log x = 0$ is off the edge of your graph, you can often bring it onto the graph by changing the units of x. For example if x is a distance ranging from 20 cm to 200 cm, the place where $\log x = 0$ is when $x = 1$ cm, which will probably be off the left side of your graph. But if we change the units of x to meters, then the place where $\log x = 0$ is where $x = 1$ m, which is right in the middle of your data.)

To estimate the uncertainties of these quantities, draw a new line with the *largest* slope that you think might be consistent with your data, and another line with the *smallest* slope consistent with your data, and find the slope and intercept for each of these lines. The greatest and least slope will then bracket the uncertainty range of n and you can use the greatest and least values of $\log k$ determine the greatest and least values of k, which bracket the uncertainty range of k.

To go further than crude estimates, one needs the help of a computer. The program *LinReg*, which is discussed in Chapter 10 of this manual, makes it very easy to plot log-log graphs and find the best-fit slope (with its uncertainty) and the best-fit intercept (and its uncertainty).
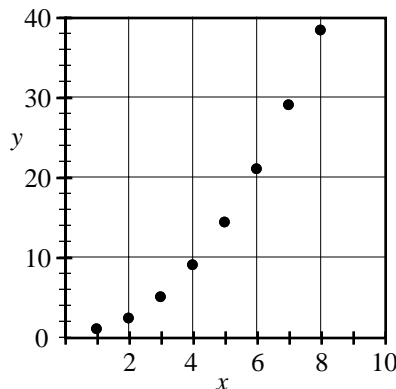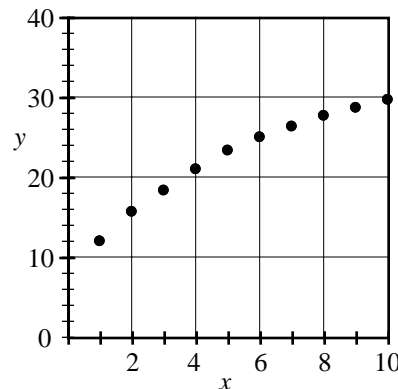

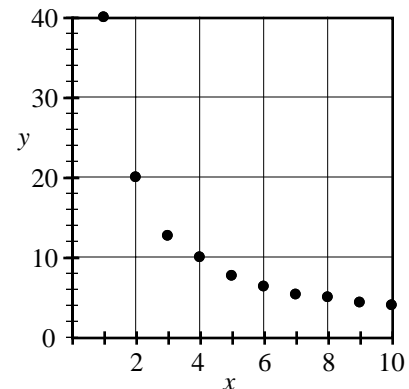**Figure 5.3:** $y = x^a$, $a > 1$.   **Figure 5.4:** $y = x^a$, $a < 1$.   **Figure 5.5:** $y = x^a$, $a < 0$.

EXERCISES

Exercise 5.1

The table to below gives the orbital periods T (in years) of the planets known to Newton as a function of their mean distance R from the sun in AUs (where 1 AU = the earth's mean orbital radius). Plot plot a log-log graph of the period versus the distance on the graph paper provided as Figure 5.6 on the next page.

**Table 5.3 Planetary Periods
vs. Mean Orbital Distances**

Planet	Distance (AU)	Period (y)
Mercury	0.39	0.24
Venus	0.72	0.62
Earth	1.00	1.00
Mars	1.52	1.88
Jupiter	5.20	11.86
Saturn	9.54	29.46

Exercise 5.2

Assuming that the period and distance are related by a power-law of the form $T = kR^n$, where n is an integer or simple fraction, what does your graph suggest is the likely value of n?
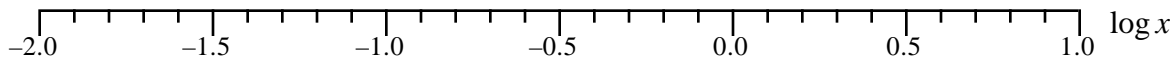
Exercise 5.3

Find the value of k (with appropriate units) for the data of Table 5.3 from the intercept of your log-log graph. Combine this with the result of exercise 5.1 to find the power-law equation (of the type given in equation 5.1) that seems to fit this data.
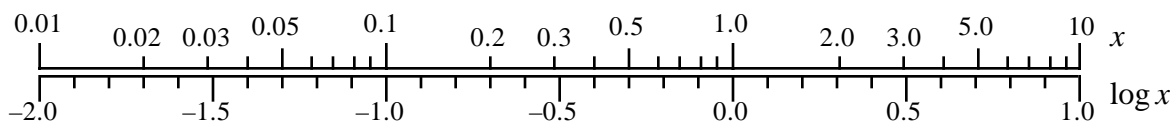
Figure 5.6

5.5 (OPTIONAL) USING LOG-LOG PAPER

If you have more than five or ten data points, calculating the logarithms quickly gets tedious even with a calculator. To reduce this tedium (which would have been particularly gruesome in the era before computers and calculators), someone invented a special kind of graph paper called **log-log paper**. In effect, this kind of graph paper calculates the logarithms for you.

Imagine that you have data for an independent variable x that ranges from, say, 0.01 m to about 10 m. The values of $\log x$ would then range from about –2.0 to 1.0, and a useful horizontal scale might look something like this:

$$-2.0 \qquad -1.5 \qquad -1.0 \qquad -0.5 \qquad 0.0 \qquad 0.5 \qquad 1.0 \quad \log x$$

Now, imagine that we were to also draw a scale immediately above this scale that showed the corresponding values of x. The two scales together would look like this:

$$0.01 \quad 0.02 \ \ 0.03 \quad 0.05 \quad 0.1 \quad 0.2 \ \ 0.3 \quad 0.5 \quad 1.0 \quad 2.0 \ \ 3.0 \quad 5.0 \quad 10 \ \ x$$
$$-2.0 \qquad -1.5 \qquad -1.0 \qquad -0.5 \qquad 0.0 \qquad 0.5 \qquad 1.0 \quad \log x$$

(Note how the pattern of the spacing between marks on the upper scale is identical for each power of 10.) Now, note that if we had graph paper that had its axes pre-labeled as shown in the *upper* scale, then we could locate points on the plot directly according to their value of x rather than having to compute the value of $\log x$ for each data point.

The way that the pattern repeats for each power of 10 makes it possible to create general and flexible graph paper with essentially pre-labeled scales. Figure 5.7 on the next page is an example of log-log paper which has 3 cycles of the pattern horizontally and vertically, making it possible to display data points whose x and/or y values span up to three powers of 10 (or three **decades**). If you compare this graph paper to the double-axis shown above, you will see that *only* the equivalent of x scale is displayed on this paper: the $\log x$ scale has been suppressed for the sake of clarity, but should be considered implicit. Also you will note that the publishers of the graph paper do not commit you to particular powers of 10: each decade is labeled as if it spans from 1 to 10. You can cross out the numbers shown to adapt the graph paper to the particular ranges of your data points (this is conventionally done for just the numbers labeled 1 or 10 on the graph paper). Figure 5.7 shows how you would do this for the planetary orbit data given in Table 5.3.

The point is that with just a little relabeling you can use graph paper like this to construct quickly a log-log graph without having to do any actual calculations of logarithms. This is great for doing low-level, quickie graphs of a set of data that you think might reflect a power-law relation. $y = kx^n$. You can even read the value of k directly from the graph by finding the point where your best-fit line crosses the vertical line corresponding to $x = 1$ unit and reading the vertical coordinate of this point according to the vertical axis.

But how can you compute the slope n of data drawn on such graph? Remember each axis has an implicit *linear* scale that reflects the log of the value displayed. "Linear" means that the change in the value of $\log x$ or $\log y$ is proportional to the physical distance on the sheet of paper. So to find the slope, all that you have to do is measure the rise of your line (in cm on the sheet of paper!) and divide by the run (in cm).

Using log-log paper is optional in this course. But you can purchase sheets of log-log paper from Connie (the department secretary) for 10¢ per page if you would like to use it for quick low-level graphing.

(OPTIONAL) Exercise 5.4

Plot the planetary orbit data in Table 5.3 on the log-log paper shown below, and use the methods described in the previous section to find k and n assuming that $T = kR^n$. Check that these agree with the values you found before.

Figure 5.7: An example of 3-cycle $\times$ 3-cycle log-log paper.