9

PROPAGATION OF UNCERTAINTY

"She drew up plans of economy, she made exact calculations…"
--- *Persuasion*

9.1 INTRODUCTION

In many kinds of physics experiments, one would like to know the uncertainty in a quantity (call it f) that is *calculated* from directly measured and uncertain quantities $a, b, c, \ldots$, that is, f is a *function* $f(a,b,c,\ldots)$ of the measured quantities $a, b, c, \ldots$. For example, this problem would arise in an experiment where we want to determine the uncertainty in an object's speed if that speed is calculated from uncertain time and distance measurements The general problem of determining a calculated quantity's uncertainty is called the problem of **propagation of uncertainties** (expressing the idea that uncertainties in measured quantities beget uncertainties in quantities calculated from them). The goal of this chapter is to explore means for intelligently addressing this problem.

9.2 SOME NOTATION AND TERMINOLOGY

We will use the symbol $U[f]$ to refer to the experimental uncertainty in any quantity f, whether that uncertainty has been directly measured or calculated from the uncertainties in other measured quantities. If the quantity f depends on measured quantities $a, b, \ldots$, then its uncertainty $U[f]$ should be related to the uncertainties in $a, b, \ldots$ (that is, on $U[a]$, $U[b]$, and so on) in some way that we *should* be able to calculate knowing how f depends on these variables.

It turns out that the most useful quantity to know when dealing with the problem of propagation of uncertainty is a variable's **fractional uncertainty**, which is defined to be ratio of the variable's uncertainty to its measured or best guess value:

$$\text{fractional uncertainty of } f \equiv Q[f] \equiv \frac{U[f]}{f} \tag{9.1}$$

(The symbol Q is meant to make you think "quotient."). This quantity is very closely related to the concept of *percent uncertainty*: to get the percent uncertainty from the fractional uncertainty, simply multiply by 100. These ideas are so closely and simply related that we will often treat "fractional uncertainty" and "percent uncertainty" as if they were the same.

As an example, say that the measured value of f is (5.96± 0.60) cm. The *fractional uncertainty* of f is then $Q[f]$ = 0.6 cm/5.96 cm = 0.10, and its *percent uncertainty* is 10% (that is, its uncertainty is equal to 10% of its best-guess value).

Note that whatever units a quantity f might have, $U[f]$ has the *same* units, so the ratio $Q[f]$ of these quantities (and thus the fractional or percent uncertainty) will always be a *unitless* number. This observation is also a reminder that a quantity's uncertainty $U[f]$ and its fractional uncertainty $Q[f]$, while related, are *not* the same thing (if they were, they would have the same units).

9.3 A GENERAL APPROACH TO PROPAGATION OF UNCERTAINTIES

Think of the function $f(a,b,c,\ldots)$ as a machine that has a handle (like a control stick) corresponding to each of its input variables $a, b, c, \ldots$ and a big dial with a pointer that indicates the output value f. Each of the input variables affects the final value shown on the dial, so adjusting the positions of the handles individually or in combination will change the value shown on the dial.

Now if the input value a has an uncertainty $U[a]$, then we can wiggle the handle corresponding to the variable a back and forth from its most probable value a by a positive or negative amount δa in the range $|\delta a| \le U[a]$ and still be consistent with the experimental data. This wiggling will cause the value of f indicated by the dial to wiggle back and forth from its central value by a certain amount as well. Let us define δf_a to be the (presumably small) *change* in the value of f from its central when a is moved from its central value by $\delta a_{\max} = +U[a]$ (corresponding to the upper extreme limit of a's uncertainty range), while the other handles are held constant. Similarly, let δf_b be the change in f when b is moved from its central value by an amount $\delta b_{\max} = +U[b]$ while the other variables are held constant, and so on.

Now, what is the uncertainty in f when all of its variables are free to wiggle around within their uncertainty ranges simultaneously? The *maximum* distance f could be from its central value is $\delta f_{\max} = |\delta f_a| + |\delta f_b| + |\delta f_c| + \ldots$ if all of the input values happen to be simultaneously at whichever edge of their uncertainty range causes them to shift f in the same direction. But this is fairly unlikely, because there is only roughly a 5% chance that any *single* variable will be at or beyond either limit of its uncertainty range; the likelihood that *all* of the variables are simultaneously at or beyond their limits on the correct side to push the value of f in the same direction is vanishingly small.

It turns out that because of this, a statistically more accurate estimate of the uncertainty of f due to the uncertainties in all of its variables is

$$U[f] \approx \sqrt{\delta f_a^2 + \delta f_b^2 + \delta f_c^2 + \ldots} \tag{9.1}$$

(The proof is beyond our scope here.) Quantities whose effects are "added" by squaring, adding, and then taking the square root like this are said to be **added in quadrature**. Calculating $U[f]$ thus reduces to the problem of finding the changes $\delta f_a, \delta f_b, \ldots$ due to each variable separately.

This can be done easily in simple cases. Consider the special case where $f(a,b) = a - b$. If we increase a to $a + \delta a$ while keeping b fixed, then f changes to

$$f + \delta f = a + \delta a - b \quad \Rightarrow \quad \delta f = \delta a \quad \Rightarrow \quad \delta f_a = \delta a_{\max} = +U[a] \tag{9.2}$$

after subtracting $f = a - b$ from both sides. Similarly, you can easily see that $\delta f_b = -U[b]$ (negative because when b goes up, f goes down). Therefore the total uncertainty in f in this case is

$$U[f] = \sqrt{(U[a])^2 + (U[b])^2} \qquad \text{when } f = a - b \tag{9.3}$$

Therefore, if a and b have the same uncertainty then the best estimate of the uncertainty in f is not $2U[a]$ (as one might naively expect) but rather $\sqrt{2}U[a] \approx 1.4U[a]$.

On the other hand, if $U[a]$ is more than about 3 times larger than $U[b]$, then $(U[a])^2$ is more than 9 times larger than $U[b]$, and thus dominate the expression for $U[f]$ in equation 9.3. This is a specific example of a more general feature of the original equation 9.1: that equation implies that the uncertainty $U[f]$ is typically dominated by *one* variable, the variable whose variation over its uncertainty range causes the greatest change in the value of f.

9.4 THE WEAKEST-LINK RULE

The majority of calculated quantities f that arise in physics experiments can be put in the form

$$f(a,b,c,\ldots) = k a^m b^n c^j \ldots \tag{9.4}$$

where k is a constant and m, n, and j are exponents that may be positive or negative (these exponents are usually integers or simple fractions). A dependence of this form on the variables $a, b, c, \ldots$

is called a **power-law** dependence. For example, an object's calculated speed v depends on the distance D it had to travel and the time T that it took to travel that distance according to the power-law relation $v = D/T = D^1T^{-1}$ ($k = 1$ here).

If equation 9.4 is true, then the **weakest-link rule** provides a fast and simple way of estimating the uncertainty $U[f]$ in the calculated quantity f:

> The fractional uncertainty $Q[f]$ of $f = ka^m b^n c^j \ldots$ is approximately equal to the *largest* of the values $|m|Q[a]$, $|n|Q[b]$, $|j|Q[c]$, and so on.

The fractional uncertainties $Q[a]$, $Q[b]$, $Q[c]$, … of the variables are typically quite different in a real experiment, and so doing a few rough divisions in your head can quickly guide you to the variable whose fractional uncertainty is largest.

We will look at *why* this rule is correct in a moment. First note that this rule says two interesting things. The first is that the "weakness" (that is, the fractional uncertainty) of a calculated quantity f is determined *primarily* by the "weakest" of the quantities on which it depends (the "weakest" here being the quantity whose fractional uncertainty times its exponent is largest). This is fundamentally because of the effect noted in the last paragraph of the last section: one variable's effect typically dominates the sum in equation 9.1. The rule's name emphasizes this by bringing to mind the old saying "the strength of a chain is determined by its weakest link." (Note that contrary to the currently popular game show, the "weakest link" is the one we *keep* in our calculation!)

The second interesting thing that the rule says is that it is the *fractional* uncertainty in f that is related in a simple way to the *fractional* uncertainties of its variables. This was *not* the case when we are talking about the simple sum or difference of variables (as equation 9.3 shows), but as we will shortly see, it is the most natural way to deal with power-law relations.

Let's see how this rule might work in a given situation. Imagine that we are computing the magnitude of an object's average velocity $v_{\text{avg}} = D^1T^{-1}$, where D is the distance it travels during a time T. Say that we have measured $D = (12.12 \pm 0.02)$ m and $T = (0.82 \pm 0.05$ s). The best guess value of $v_{\text{avg}} = 12.12 \text{ m} / 0.82 \text{ s} = 14.8$ m/s. The fractional uncertainties in D and T are:

$$Q[D] = \frac{U[D]}{D} = \frac{0.02 \text{ m}}{12.12 \text{ m}} = 0.0017, \qquad Q[T] = \frac{0.05 \text{ s}}{0.82 \text{ s}} = 0.061 \tag{9.6}$$

The fractional uncertainty in T is more than 35 times larger than that for D so it dominates. According to the weakest link rule, the fractional uncertainty in $v_{\text{avg}} = D^1T^{-1}$ is thus given by

$$Q[v_{\text{avg}}] \approx |-1|\, Q[T] = 0.061 \quad \Rightarrow \quad U[v_{\text{avg}}] = 0.061 v_{\text{avg}} = 0.061(14.8 \text{ m/s}) = 0.9 \text{ m/s} \tag{9.7}$$

Note that the calculations here are quick and simple: that is the beauty of the weakest-link rule.

The general "proof" of the weakest-link rule is somewhat beyond our mathematical means here, but let's see how we might "prove" it in the simple case where $f = f(a,b) = ka^2b$. If b changes to $b + \delta b$ while a remains the same, then f changes to

$$f + \delta f_b = ka^2(b + \delta b) = f + ka^2\delta b \quad \Rightarrow \quad \delta f_b = ka^2 \delta b \tag{9.8a}$$

If we now divide both sides of this by $f = ka^2b$ and set $\delta b = +U[b]$, we find that

$$\frac{\delta f_b}{f} = \frac{ka^2\delta b}{ka^2 b} = \frac{\delta b}{b} = \frac{U[b]}{b} \equiv Q[b] \tag{9.8b}$$

If we change a to $a + \delta a$ while b remains the same, then f changes to $f + \delta f_a = k(a + \delta a)^2 b$.

Writing out the square and subtracting $f = ka^2b$ from both sides, we get:

$$f + \delta f_a = k(a^2 + 2a\delta a + \delta a^2)b = f + 2kab\delta a + kb(\delta a)^2$$

$$\Rightarrow \quad \delta f_a = 2kab\delta a + kb(\delta a)^2 \tag{9.9a}$$

Dividing both sides of the result by $f = ka^2b$ yields:

$$\frac{\delta f_a}{f} = \frac{2kab\delta a}{ka^2b} + \frac{kb(\delta a^2)}{ka^2b} = 2\frac{\delta a}{a} + \left(\frac{\delta a}{a}\right)^2 \tag{9.9b}$$

Now, if we can assume that the variation δa due to a's uncertainty is much smaller than the value of a itself, then $(\delta a / a)^2 << \delta a / a$, and we can ignore the second term in equation 9.9b in comparison to the first. Then, if we set $\delta a = U[a]$ we find that

$$\frac{\delta f_a}{f} = 2\frac{U[a]}{a} = 2Q[a] \tag{9.9c}$$

If we now divide both sides of equation 9.1 by f and substitute in the results of equations 9.8b and 9.9c, we find that

$$Q[f] \equiv \frac{U[f]}{f} = \frac{1}{f}\sqrt{\delta f_a^2 + \delta f_b^2} = \sqrt{\left(\frac{\delta f_a}{f}\right)^2 + \left(\frac{\delta f_b}{f}\right)^2} = \sqrt{(2Q[a])^2 + (Q[b])^2} \tag{9.10}$$

Whichever of $2Q[a]$ or $Q[b]$ is even marginally larger than the other will dominate inside the square root and thus be essentially equal to $Q[f]$. Thus we have seen that the weakest link rule does indeed adequately summarize the more exact calculation in this case as long as (1) the fractional uncertainty in a is fairly small (so that we can ignore the complicating term in equation 9.9b), and (2) one of $2Q[a]$ or $Q[b]$ is at least somewhat larger than the other.

9.5 WHAT IF THE WEAKEST-LINK RULE DOESN'T APPLY?

The weakest-link rule does *not* apply to situations where f's dependence on its variables is not a power-law relation (for example, the simple sum $f(a,b) = a+b$). The weakest-link rule is also not very accurate in situations where the fractional uncertainties in the variables are large fractions of 1, or when two fractional uncertainties are nearly the same. What do we do in such situations?

The first level of approximation is to ***use the weakest link rule anyway,*** and simply recognize (and state in your lab notebook) that the estimate of the uncertainty might well be inaccurate. The weakest-link rule will almost always yield estimates good to within a factor of two or so unless your formula for f involves logarithms or exponentials. In situations where one is not interested in high degree of precision this may be acceptable, as long as you *recognize* situations when the rule might not be expected to give accurate results and factor that into your conclusions.

A better way to determine the uncertainty of f would be to ***calculate many values*** of f using values of its variables $a, b, c, \ldots$ that are randomly chosen from the raw data for these variables. Then one can determine the uncertainty of f in the usual way by evaluating the standard deviation of the set of values for f and so on. (This is essentially what we did in the speed of sound lab: we computed speeds for many different time and distance values.) This method almost *always* gives an excellent estimate of $U[f]$ as long as the number of values of f that you generate is reasonably large (more than 20 at least!). However, because this method is *so* tedious, we cannot recommend it unless you have a computer program to do the work.

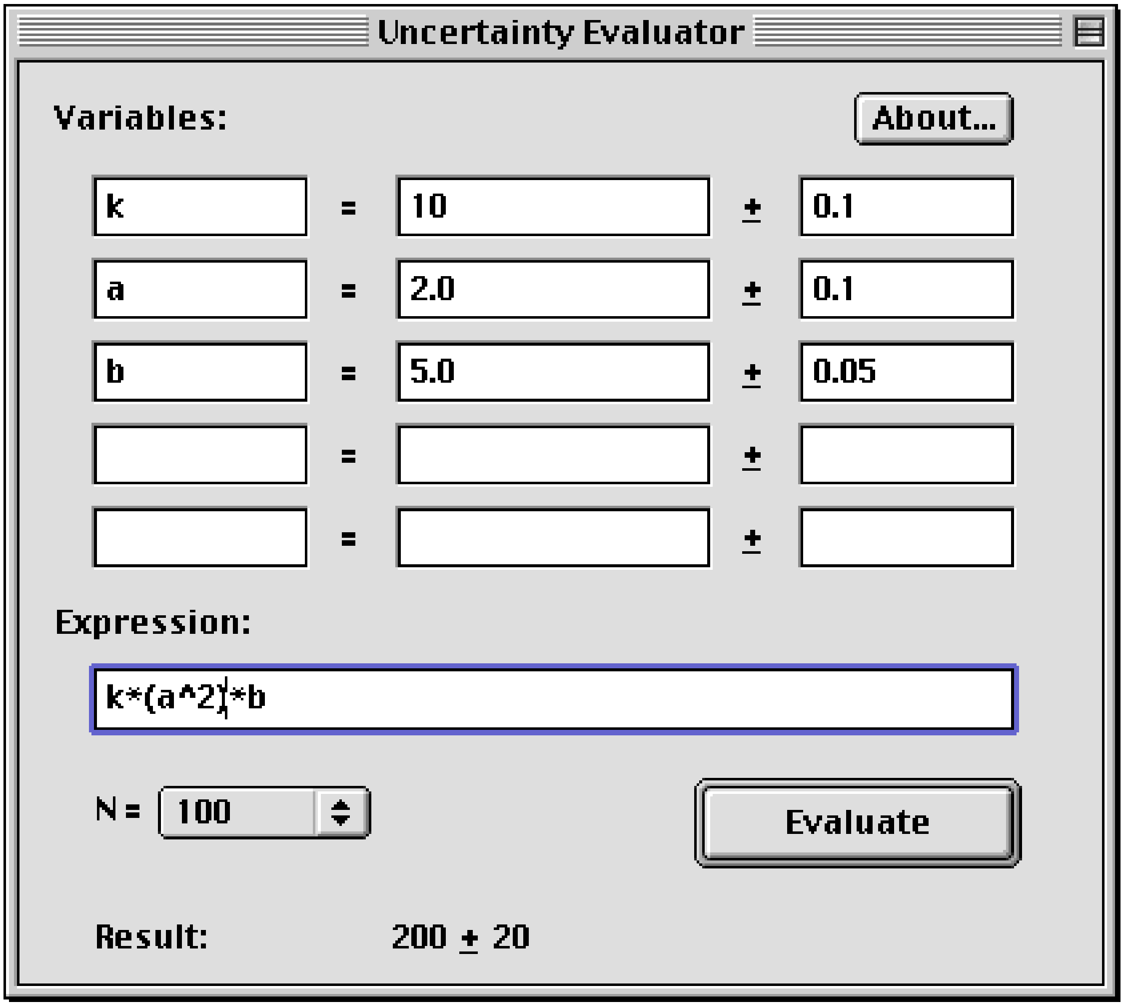

Figure 9.1: The *PropUnc* program.

An approach of last resort is to ***apply the general method*** outlined in section 9.2. Calculate (by hand) the variation in f when you vary each variable from its central position to the upper edge of its uncertainty range while leaving the other variables constant. Then use equation 9.1 to compute the total uncertainty in f from these individual variations. This will generally be pretty tedious compared to the weakest link method, but does yield reasonably accurate answers in *all* cases. This is the method that you *must* use if f involves a logarithm or exponential (unless you have a computer program that can do the calculation outlined in the previous paragraph).

Of course, if f involves the simple sum or difference of two variables, one can apply equation 9.3, which we derived especially for the simple difference case. (You should be able to convince yourself that equation 9.3 also applies to the case of a simple sum.)

9.6 THE *PropUnc* PROGRAM

PropUnc is a computer program that uses the "calculate many values" approach to generate an accurate value of uncertainty of f in *all* cases involving five or fewer variables. A screen shot of the program set up to calculate the uncertainty of the function $f = ka^2b$ is shown in Figure 9.1. All that you have to do to use the program is type symbols, values, and uncertainties for your basic variables in "variables" section and the symbolic expression for f in the "expression" section and punch the "Evaluate" button. The program then calculates a randomly-chosen value for each variable that lies within the uncertainty range you specified for that variable and calculates the value of f using randomly-perturbed variable values using the formula you supply. The computer repeats this process N times (N = 100 by default). Finally, the computer calculates the standard deviation of the N values of f it has generated and the uncertainty in f from that.

In other words, the computer simulates having N teams of experimenters like yourselves who have measured the same variables and have used them to calculate values of f. The uncertainty in the value of f is clearly related to the spread in the values obtained by the N ficticious teams.

Note that in the case shown in the figure, the fractional uncertainty in f is 10%. Note that the quantity a has by far the largest fractional uncertainty (5% compared to 1% for the other variables), so the weakest link rule would say that $Q[f] \approx 2Q[a] = 2 \cdot 5\% = 10\%$. Thus the program agrees with the weakest-link rule in this case. If you press the "Evaluate" button again, however, you may get slightly different results, because of the random nature of the simulation. Choosing larger values of N will make the calculation more accurate, but could be slow on a old computer.

We actually would rather you use the weakest-link rule whenever you can; you will not always have *PropUnc* handy in real life, so it is good to practice using the weakest-link rule, which is simple and usually gives good results. You *may* use *PropUnc* (1) to *check* a weakest-link calculation, or (2) whenever the weakest-link rule or equation 9.3 does not apply. *PropUnc* is installed on all the lab computers and also may be freely downloaded from the Physics 51 web site.

9.6 SOME EXAMPLES

Example 9.6.1: Suppose that you want to find the uncertainty in the volume of a cylinder when you have measured its diameter and height. The volume V of a cylinder in terms of its diameter d and height h is given by $V = \pi\frac{1}{4}d^2h$. Here the volume has power-law dependences on the variables d and h, so we should be able to apply the weakest-link rule. Suppose that our measurements are $d = (0.200 \pm 0.002)$ m and $h = (0.600 \pm 0.003)$ m. The fractional uncertainties in d and h are:

$$Q[d] = \frac{0.002\text{ m}}{0.200\text{ m}} = 0.01, \qquad Q[h] = \frac{0.003\text{ m}}{0.600\text{ m}} = 0.005 \tag{9.11a}$$

Note that even though the absolute uncertainty of d is smaller than that for h (0.002 m compared to 0.003 m), the fractional uncertainty of d is larger. Moreover, since $V \propto d^2$, the weakest link rule tells us that we should be comparing $2Q[d]$ to $Q[h]$: we see that in this case the first is four times larger than the second. Therefore, according to the weakest-link rule

$$Q[V] \approx 2Q[d] \approx 0.02 \tag{9.11b}$$

The 1% uncertainty in d thus leads to a 2% uncertainty in V. Now that we have the fractional uncertainty in V we can find the absolute uncertainty pretty easily. The central volume value that we calculate from our best-guess estimates of d and h is

$$V = \pi\tfrac{1}{4}d^2h = \frac{\pi}{4}(0.200\text{ m})^2(0.600\text{ m}) = 0.0188\text{ m}^3 \tag{9.12}$$

This value is uncertain to 2%, so its absolute uncertainty must be

$$U[V] = V \cdot Q[V] = 0.02(0.0188\text{ m}^3) = 0.000377\text{ m}^3 \approx 0.0004\text{ m}^3 \tag{9.13}$$

where I have rounded the uncertainty to one significant digit. An uncertainty this size means that it is pointless to include more digits than we already have in equation 9.12. So a statement of this value and its uncertainty would be $V = (0.0188 \pm 0.0004)$ m^3 or $(1.88 \pm 0.04) \times 10^2$ m^3. Note that in either case we have written both values so that they are multiplied by the same power of 10. This makes the values much easier to compare.

Example 9.6.2: Imagine that the number of bacteria in a certain colony at a certain time is N = 305,000 ± 15,000. What is the uncertainty in $f = \ln N$? (You might need to know the uncertainty of the logarithm if you want to draw an uncertainty bar for this data point on a log-log graph.)

Since $f = \ln N$ is not a power-law relation, we cannot use the weakest-link rule. If we can't use *PropUnc*, we can fall back on the general method. In this case, if we change N from its central value of 305,000 to the upper limit of its uncertainty range which is 320,000, the value of $\ln N$ changes from ln(305,000) = 12.6281 to ln(320,000) = 12.6761, so the change in f due to this change is δf_N = +0.0480. Since f only depends on N in this case, equation 9.1 implies that

$$U[f] = \sqrt{(\delta f_N)^2} = \left|\delta f_N\right| = 0.05 \tag{9.14}$$

where I again have rounded to one significant digit.

The result from *PropUnc* is shown in Figure 9.2. Note that in "computerese" ln N becomes "log(n)". I used a small n to distinguish it from the number N of trials the computer evaluates.

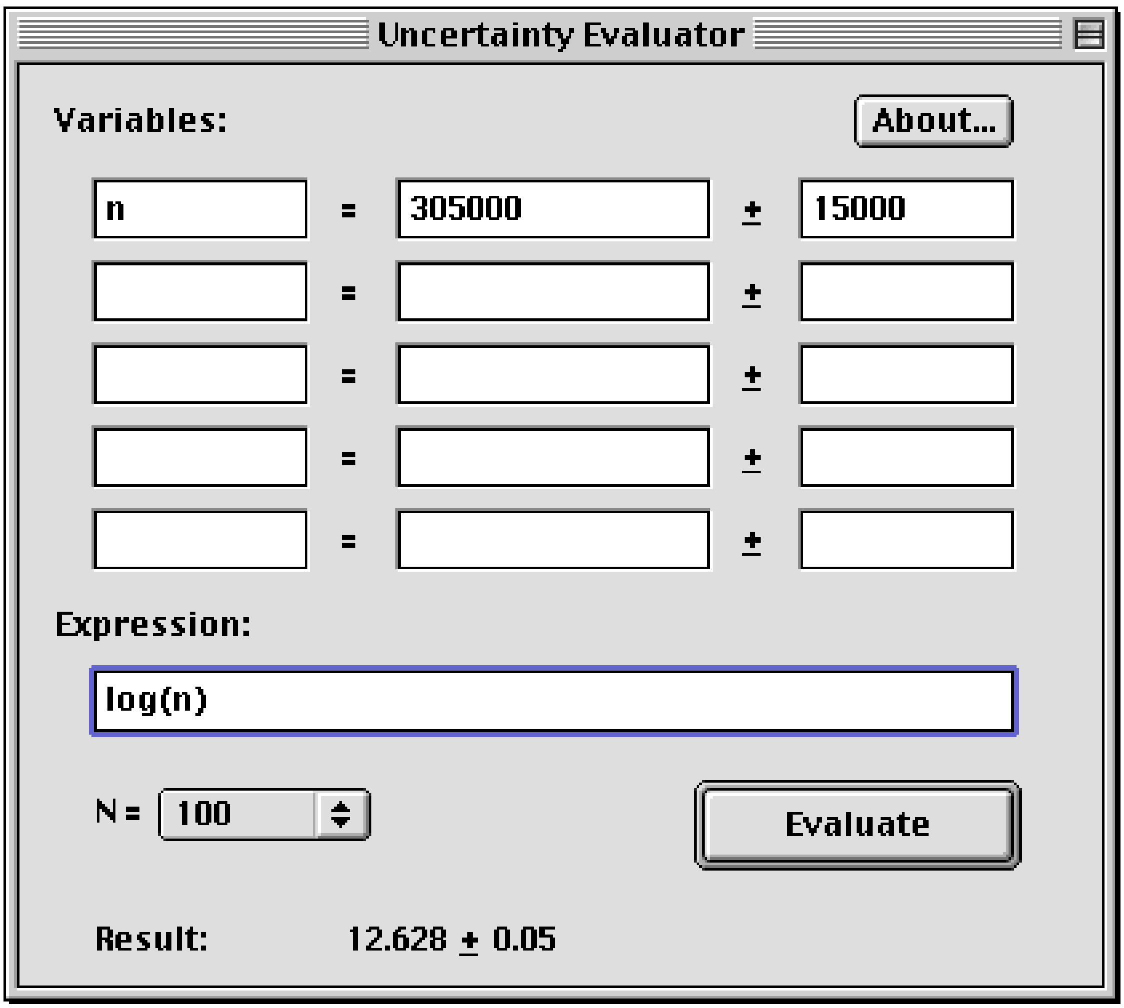

Figure 9.2: ***PropUnc*'s check of equation 9.14.**

If we were to naively apply the weakest-link rule anyway we would estimate that since the fractional uncertainty in N is 15,000 / 305,000 ≈ 0.05, the fractional uncertainty in $f = \ln N$ would also be 5%. This would lead us to estimate the uncertainty of f to be 0.05(12.63) ≈ 0.63, which is more than 10 times larger than the more correct calculation given by equation 9.14. This illustrates our earlier statement that the weakest-link rule does poorly when f involves logarithms.

9.7 THE BOTTOM LINE

You will be expected to state uncertainties of *all* calculated quantities in this lab program. Use the weakest-link rule to estimate these uncertainties whenever that rule applies; when it doesn't, either use one of the methods discussed in section 9.5 or use *PropUnc* to calculate the uncertainty.

EXERCISES

Exercise 9.1

A person is measured to run a distance of 100.00 m ± 0.05 m in a time of 11.52 s ± 0.08 s. What is the person's speed and the uncertainty of this speed according to the weakest link rule?

Exercise 9.2

A spherical balloon has a radius of 0.85 m ± 0.01 m. How many cubic meters of gas does it contain, and what is the uncertainty in your result?

Exercise 9.3

Imagine that you want to estimate the amount of gas burned by personal cars every year in the U.S. You estimate that there are an average of about 0.7 ± 0.4 cars per person in the U.S., that there are 275 million ± 30 million people in the U.S. currently, that a car is driven on the average about 15,000 mi ± 3,000 mi a year, and that the *average* number of miles per gallon that a car gets is about 23 mi/gal ± 5 mi/gal. What is the approximate amount of gas burned and what is the approximate uncertainty of this estimate?

Exercise 9.4

Equation 9.10 suggests that rather than dropping the other uncertainties entirely (as the weakest link rule suggests) perhaps we would get a more accurate estimate of the fractional uncertainty in a power-law relation by multiplying the fractional uncertainty of each variable by its power, squaring the result, adding the squares and taking the square root of the sum. Do this for the case described in Exercise 9.3 above. Is the answer you get from doing this careful way much different from just using the weakest link result? Suppose that you do some research that enables you to reduce the fractional uncertainty in the all quantities but the worst one to 1%. Does reduce the uncertainty much? If you really want to improve the uncertainty, what would be the variable to focus on, and why?